HH0093 Moment of Inertia

Now we try to derive the EOM for a rotating rigid body. Starting from EOM's for each particle,

$$\boxed{\begin{aligned} \frac{d\vec{P}_1}{dt} &= \vec{F}_1 + (\vec{f}_{12} + \vec{f}_{13} + \cdots + \vec{f}_{1N}) \\ &\ \vdots \\ \frac{d\vec{P}_N}{dt} &= \vec{F}_N + (\vec{f}_{N1} + \vec{f}_{N2} + \cdots + \vec{f}_{NN-1}) \end{aligned}}$$

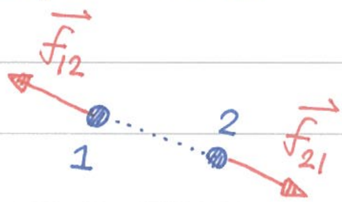
Perform the outer product $\vec{r}_i \times$ on both sides and add up the resultant equations:

$$\vec{r}_1 \times \frac{d\vec{P}_1}{dt} + \cdots + \vec{r}_N \times \frac{d\vec{P}_N}{dt} = (\vec{r}_1 \times \vec{F}_1 + \cdots + \vec{r}_N \times \vec{F}_N) + (\vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} + \cdots)$$

① Making use of $\frac{d\vec{r}_i}{dt} = \vec{v}_i \parallel \vec{P}_i$, the LHS can be written as

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i\right) = \sum_{i=1}^{N} \cancel{\frac{d\vec{r}_i}{dt} \times \vec{P}_i}^{\,0} + \vec{r}_i \times \frac{d\vec{P}_i}{dt}$$

② Suppose the internal forces satisfy $\vec{f}_{12} + \vec{f}_{21} = 0$ and $\vec{f}_{12}, \vec{f}_{21} \parallel \hat{r}_{12}$ (along the direction of $\vec{r}_{12}$),

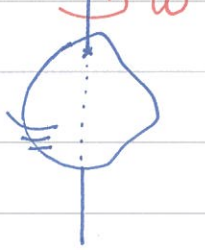


$$\vec{r}_1 \times \vec{f}_{12} + \vec{r}_2 \times \vec{f}_{21} = (\vec{r}_1 - \vec{r}_2) \times \vec{f}_{12} = 0 \;!$$

The net torque due to internal forces is ZERO. Thus, the EOM for a rotating rigid body is rather simple-looking,

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{ex}} \qquad \text{Here } \vec{\tau}_{ex} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i \,, \quad \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{P}_i$$

∅ Moment of inertia for a rigid body. Following similar steps, it's straightforward to derive the moment of inertia for a rigid body,



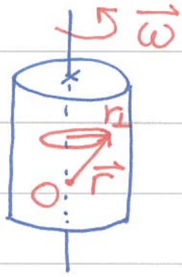
$$\boxed{I_{ij} = \sum_{\alpha=1}^{N} -m_\alpha x_{\alpha i} x_{\alpha j} + m_\alpha r_\alpha^2 \delta_{ij}}$$

Or, it can be written in matrix form,

$$\boxed{I_{ij}=\begin{pmatrix} \sum_{\alpha=1}^{N} m_\alpha(y_\alpha^2+z_\alpha)^2 & -\sum_{\alpha=1}^{N} m_\alpha x_\alpha y_\alpha & -\sum_{\alpha=1}^{N} m_\alpha x_\alpha z_\alpha \\ -\sum_{\alpha=1}^{N} m_\alpha x_\alpha y_\alpha & \sum_{\alpha=1}^{N} m_\alpha(x_\alpha^2+z_\alpha^2) & -\sum_{\alpha=1}^{N} m_\alpha y_\alpha z_\alpha \\ -\sum_{\alpha=1}^{N} m_\alpha x_\alpha z_\alpha & -\sum_{\alpha=1}^{N} m_\alpha y_\alpha z_\alpha & \sum_{\alpha=1}^{N} m_\alpha(x_\alpha^2+y_\alpha^2) \end{pmatrix}}$$

It can be a horrible job to compute I_{ij} — not only it is a tensor with 6 components ⋯ each component involves a sum!

In the following, we consider the simplest type of rotation — the rotation axis is fixed and $\vec{\omega} \parallel \vec{L}$. For simplicity, let's set the axis to be the z axis so that $\vec{\omega}=(0,0,\omega)$ and $\vec{L}=(0,0,L)$. The relation between angular velocity and angular momentum becomes



$$L_z = I_{zz}\,\omega_z \longrightarrow \boxed{L = I\omega}$$

Here the moment of inertia is

$$I = I_{zz} = \sum_{\alpha=1}^{N} m_\alpha r_{\perp\alpha}^2$$

In the continuous limit, the summation can be replaced by integral,

$$\boxed{I = \sum_{\alpha=1}^{N} m_\alpha r_{\perp\alpha}^2 = \int r_\perp^2\, dm}$$

$r_\perp = \sqrt{x^2+y^2}$ is the distance to the axis.

In this limit, the equation of motion also simplifies,

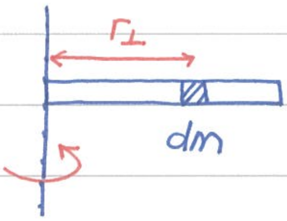
$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} \Rightarrow \boxed{\tau = I\alpha}$$

special case, not the general form!

The kinetic energy also takes a simpler form

$$K = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{1}{2}\omega_i I_{ij}\omega_j = \frac{1}{2} I_{zz}\omega_z^2 \rightarrow \boxed{K = \frac{1}{2} I\omega^2}$$

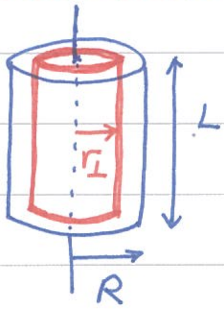
example 1. Consider a uniform rod of mass M and length L. The linear density is $\lambda = M/L$.



$$I = \int r_\perp^2 dm = \int_0^L x^2 \lambda dx$$

$$I = \frac{\lambda}{3} x^3 \Big|_0^L = \frac{1}{3}\lambda L^3 \longrightarrow \boxed{I = \frac{1}{3} M L^2}$$

example 2. Consider a uniform cylinder of mass M. The density $\rho = M/V = M/(\pi R^2 L)$.

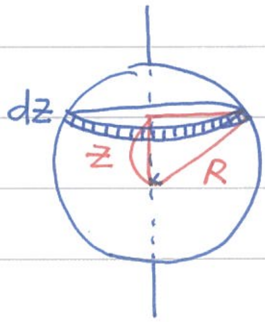


$$I = \int r_\perp^2 dm = \int_0^R r^2 (2\pi r L \cdot dr)\rho$$

$$= 2\pi\rho L \int_0^R r^3 dr = 2\pi\rho L \cdot \frac{1}{4} R^4 = \frac{\pi}{2}\rho L R^4.$$

The moment of inertia for a cylinder is $\boxed{I = \frac{1}{2} M R^2}$

example 3. Consider a uniform sphere of mass M. The density is $\rho = M/V = 3M/4\pi R^3$. Let's focus on the infinitesimal disk of radius $\sqrt{R^2 - z^2}$



$$dI = \frac{1}{2} dm \cdot (R^2 - z^2) = \frac{1}{2}\rho \cdot \pi (R^2 - z^2) \cdot dz\, (R^2 - z^2)$$

$$I = \int dI = \frac{1}{2}\pi\rho \int_{-R}^{R} (R^2 - z^2)^2 dz$$

$$= \frac{1}{2}\pi\rho \int_{-R}^{R} (R^4 + z^4 - 2R^2 z^2)\, dz$$

$$\longrightarrow I = \frac{1}{2}\pi\rho \left(2R^5 + \frac{2}{5}R^5 - \frac{4}{3}R^5\right) = \frac{1}{2}\pi\rho \cdot \frac{16}{15} R^5 = \frac{8}{15}\pi\rho R^5$$

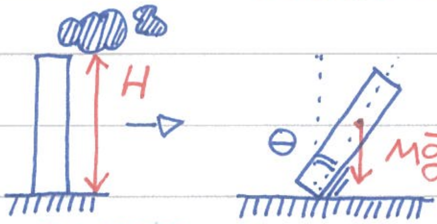
Finally, the moment of inertial for a uniform sphere is

$$\boxed{I = \frac{8}{15}\pi\rho R^5 = \frac{2}{5} M R^2} \longrightarrow I = (\text{geo-factor}) \times M \times (\text{length})^2$$

⊘ Rotational dynamics – fixed axis.

Let us apply the simplified version of EOM, $\tau = I\alpha$ to simple rotational motions.

example 1. Falling chimney. Choose the base of chimney as the reference point. The torque is $\frac{1}{2}HMg\sin\theta$. We already computed $I = \frac{1}{3}MH^2$ before.



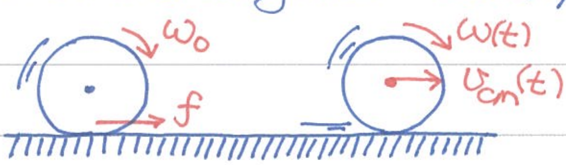
EOM $\tau = I\alpha$ → $\frac{1}{2}MgH\sin\theta = \left(\frac{1}{3}MH^2\right)\cdot\alpha$

The angular acceleration $\alpha = \frac{3g}{2H}\sin\theta$ ← not a constant.

The acceleration at the end of the chimney is the largest,

$a_{end} = H\cdot\alpha = \frac{3}{2}g\sin\theta$ ← The end acceleration can be larger than g. Why?

example 2 Rolling disk. At $t=0$, the c.m. is at rest and the angular velocity is ω_0. Choose c.m. as the reference point. The EOMs are



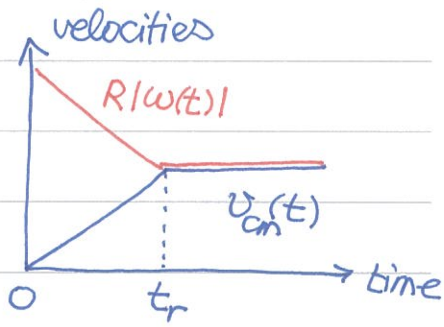
$$\boxed{\begin{aligned} Ma_{cm} &= f \\ I\alpha &= fR \end{aligned}}$$

← cm motion

← rotation.

It is easy to find the solution for $v_{cm}(t)$ and $\omega(t)$.

$v_{cm}(t) = \frac{f}{M}t$, $\omega(t) = -\omega_0 + \left(\frac{fR}{I}\right)t$



The velocity at the contact point is

$$\boxed{v_g = v_{cm} + R\omega = -R\omega_0 + f\left(\frac{1}{M} + \frac{R^2}{I}\right)t}$$

When $t = t_r$, the contact velocity is zero and the friction disappears. Let us find out t_r.

The moment of inertia for a disk is $I=\frac{1}{2}MR^2$.

$$v_g=0 \longrightarrow -R\omega_0+\frac{3f}{M}t_r=0 \qquad t_r=\frac{MR\omega_0}{3f}$$

At later time $t>t_r$, both v_{cm} and ω are constant,

$$\boxed{v_{cm}(t>t_r)=\frac{1}{3}R\omega_0\,,\quad \omega(t>t_r)=-\frac{1}{3}\omega_0}$$

The initial kinetic energy is $K_i=\frac{1}{2}I\omega_0^2=\frac{1}{4}MR^2\omega_0^2$

The final kinetic energy contains two parts: $K_f=\frac{1}{2}Mv_{cm}^2+K_{in}=\frac{1}{2}Mv_{cm}^2+\frac{1}{2}I\omega^2$

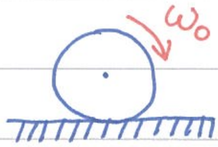
$$\longrightarrow K_f=\frac{1}{2}M\left(\frac{1}{3}R\omega_0\right)^2+\frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(-\frac{1}{3}\omega_0\right)^2=\frac{1}{12}MR^2\omega_0^2$$

According to the E-conservation derived before,

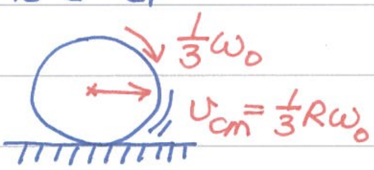
$$\boxed{\Delta\left(\frac{1}{2}Mv_{cm}^2+E_{in}\right)=W_{nc}}$$

$v_{cm}(0)=0$

$E_{in}(0)=\frac{1}{4}MR^2\omega_0^2$



At $t=t_r$



$\frac{1}{2}Mv_{cm}^2(t_r)=\frac{1}{18}MR^2\omega_0^2 \longleftarrow$ cm motion

$E_{in}(t_r)=\frac{1}{36}MR^2\omega_0^2 \longleftarrow$ rotational energy.

$$W_{nc}=\Delta\left(\frac{1}{2}Mv_{cm}^2+E_{in}\right)=\left(\frac{1}{18}+\frac{1}{36}-\frac{1}{4}\right)MR^2\omega_0^2=-\frac{1}{6}MR^2\omega_0^2$$

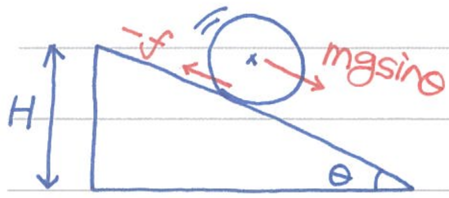
One can check the answer by computing W_{nc} directly ☺

$$W_{nc}=\int_0^{t_r}\vec{f}\cdot\vec{v}_g\,dt=-\int_0^{t_r}dt\,f\cdot\left(R\omega_0-\frac{3f}{M}t\right)$$

$$=-fR\omega_0t_r+\frac{3}{2}\frac{f^2}{M}t_r^2=\left(-\frac{1}{3}+\frac{1}{6}\right)MR^2\omega_0^2=-\frac{1}{6}MR^2\omega_0^2$$

It is rather interesting to observe that W_{nc} is independent of the friction f! To transform the rotational energy into the rolling form, the energy cost is universal.

example 3 Rolling down the hill. Assume rolling without slipping, i.e. $v_g = 0$.



EOM's :
$$\boxed{\begin{aligned} Mg\sin\theta - f &= Ma_{cm} \\ -fR &= I\alpha \end{aligned}}$$

Because $v_g = v_{cm} + R\omega = 0$, $a_{cm} + R\alpha = 0$.

$$-fR = I\alpha \rightarrow f = \frac{-I\alpha}{R} = \left(\frac{I}{R^2}\right)\cdot a_{cm} \leftarrow \text{substitute into EOM.}$$

$$Mg\sin\theta = \frac{I}{R^2}a_{cm} + Ma_{cm} = (M + I/R^2)\,a_{cm}$$

Finally, the acceleration of c.m. is $\boxed{a_{cm} = g\sin\theta\left(\frac{1}{1+I/MR^2}\right)}$

Suppose the system is at rest initially.

$$v_{cm} = a_{cm}t,\quad x_{cm} = \frac{1}{2}a_{cm}t^2 \rightarrow \boxed{\frac{1}{2}Mv_{cm}^2 = \frac{MgH}{1+I/MR^2}} < MgH.$$

Let's check the E-conservation again.

$$\Delta\left(\frac{1}{2}Mv_{cm}^2 + U_{ex} + E_{in}\right) = W_{nc}$$

$$\frac{1}{2}Mv_{cm}^2(0) = 0 = E_{in}(0)$$

$$U_{ex}(0) = MgH$$

After rolling down the height H,

$$E_{in} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{I}{R^2}v_{cm}^2 = \frac{1}{2}Mv_{cm}^2\cdot\frac{I}{MR^2} = MgH\frac{I/MR^2}{1+I/MR^2}$$

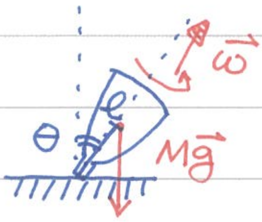
$$\rightarrow \boxed{\frac{1}{2}Mv_{cm}^2 + E_{in} = MgH\frac{1+I/MR^2}{1+I/MR^2} = MgH}$$

The work done by the friction is

$$W_{nc} = \Delta\left(\frac{1}{2}Mv_{cm}^2 + U_{ex} + E_{in}\right) = MgH - MgH = 0\ !$$

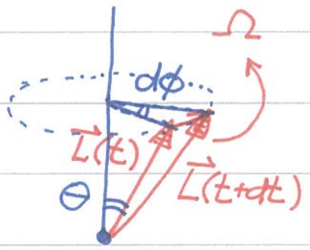
Why is the work done by friction ZERO here? The potential energy transforms into rolling form $\left(\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2\right)$ without any penalty. ☺

① Precession of a rotating top. Suppose the distance from cm to the reference point is ℓ



$$\boxed{\tau = mg\ell\sin\theta}$$

Now we need to find out the change of angular momentum in pression:



$$|d\vec{L}| = L\sin\theta \cdot d\phi = L\sin\theta\,\Omega\,dt$$

$$\rightarrow \boxed{\left|\frac{d\vec{L}}{dt}\right| = L\Omega\sin\theta}, \quad \text{here } L = I\omega.$$

From the EOM $\vec{\tau} = d\vec{L}/dt$,

$$mg\ell\cancel{\sin\theta} = L\Omega\cancel{\sin\theta} \rightarrow \boxed{\Omega = \frac{mg\ell}{L} = \frac{mg\ell}{I\omega}}$$

Thus, we find that $\Omega\omega = mg\ell/I$ = const as discussed in previous lecture ☺ It is important to notice that Ω is independent of the tilting angle. θ.

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