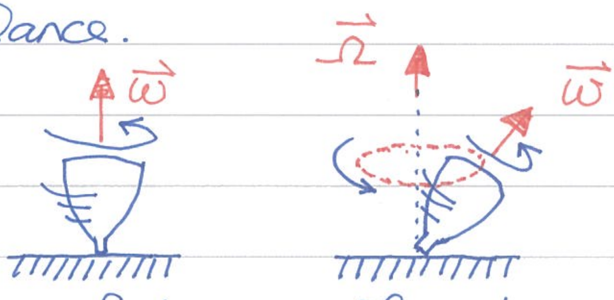
HH0092 How Does a Rotating Top Maintain its Balance?

As demostrated in class, rotation seems to be of crucial importance for a spinning top to maintain its balance.

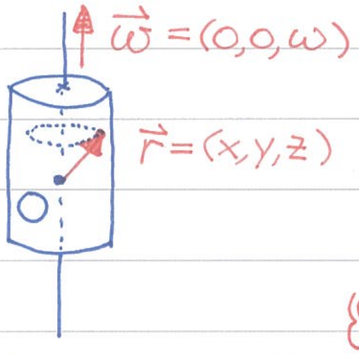
$$\boxed{\Omega \cdot \omega = \text{const}}$$

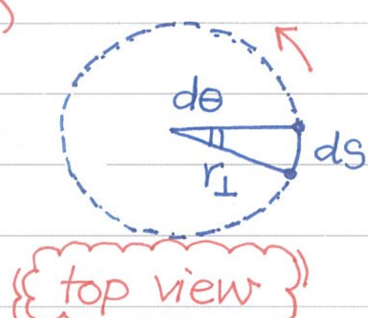
angular velocity for precession



Why is $\omega \neq 0$ so important for robust balance? Why does precession speed up when the spinning top slows down? Let us start with the rotation of a rigid body.

① Rotation around a fixed axis. Consider a rigid body rotating around the z axis. A point in the rigid body just traces out a circular motion with speed v.





$$\boxed{\begin{aligned} v &= \frac{ds}{dt} = r_\perp \frac{d\theta}{dt} \\ &= r_\perp \omega \end{aligned}}$$

where $r_\perp = \sqrt{x^2+y^2}$ is the distance to the rotation axis. The angular velocity $\omega = d\theta/dt$ is the same for all points in the rigid body even though their velocities are different. We can write the relation in vector form,

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}} \longrightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-\omega y, \omega x, 0) = \vec{v}$$

note that $\vec{v} \cdot \vec{r} = 0$ and $\vec{v} \cdot \vec{\omega} = 0$

, thus $|\vec{v}| = r_\perp \omega$ yes!

Taking the time derivative to find $\vec{a}(t)$:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

$$\longrightarrow \boxed{\vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v}}$$

angular acceleration $\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$

Because the rotational axis is fixed, $\vec{\alpha} \parallel \vec{\omega}$. In consequence,

<u>tangent</u> $\vec{a}_t = \vec{\alpha}\times\vec{r}$ $\qquad a_t \equiv |\vec{a}_t| = r\alpha\sin\theta = r_\perp\alpha$

<u>normal</u> $\vec{a}_n = \vec{\omega}\times\vec{v}$ $\qquad a_n \equiv |\vec{a}_n| = |\vec{\omega}\times(\vec{\omega}\times\vec{r})| = r_\perp\omega^2$

It is important to emphasize that the above results are valid only if the rotational axis is fixed ☺

⊘ Rotational EOM. To describe the rotational dynamics, let us start with the single-particle system.

$$\vec{F} = \frac{d\vec{p}}{dt} \longrightarrow \vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt}$$

Note that $\frac{d\vec{r}}{dt}\times\vec{p} = 0$, we can rewrite EOM

$$\vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt} + \underbrace{\frac{d\vec{r}}{dt}\times\vec{p}}_{\text{zero}} = \frac{d}{dt}(\vec{r}\times\vec{p})$$

Thus, it is natural to introduce two vector quantities,

<u>torque</u> $\boxed{\vec{\tau} \equiv \vec{r}\times\vec{F}}$ $\qquad$ <u>angular momentum</u> $\boxed{\vec{L} \equiv \vec{r}\times\vec{p}}$

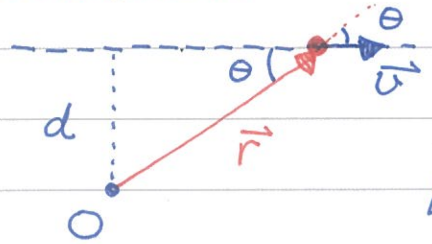
The EOM for rotational dynamics can be written as

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

(Compared with $\vec{F} = \frac{d\vec{p}}{dt}$, quite similar)

If the torque is zero (not necessarily $\vec{F} = 0$), the angular momentum is conserved ☺

example 1. linear motion. Choose a reference point O with distance d to the straight line.

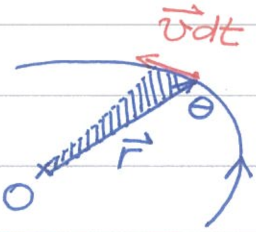


$$L = |\vec{r}\times\vec{p}| = r\cdot mv\cdot\sin\theta = mvd$$

As long as the velocity is constant, the angular momentum is also constant. But its magnitude depends on O.

example 2. Kepler's 2^{nd} law.

Between t and t+dt, the planet sweeps over area dA



$$dA = \frac{1}{2}(vdt)\cdot r\cdot\sin\theta \longrightarrow \boxed{\frac{dA}{dt} = \frac{1}{2}rv\sin\theta}$$

Because the gravitation force is along the radius direction,

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = 0 \;!!$$

It implies that $\vec{L}$ is conserved, i.e. $|\vec{L}| = mvr\sin\theta = \text{const.}$

$$\longrightarrow \boxed{\frac{dA}{dt} = \frac{L}{2m} = \text{const}} \longleftarrow \text{Kepler's } 2^{nd} \text{ law.}$$

(II) Angular velocity and angular momentum.

Compare translational and rotational motions

$$\vec{F} = \frac{d\vec{p}}{dt} \longleftrightarrow \vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{p} = m\vec{v} \longleftrightarrow \boxed{\vec{L} = I\vec{\omega}}\;?$$

Is it true that $\vec{L}$ and $\vec{\omega}$ are always parallel?

Let us concentrate on single-particle system first and try to find the relation between $\vec{L}$ and $\vec{\omega}$.

Start with the velocity $\vec{v} = \vec{\omega} \times \vec{r}$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y,\ \omega_z x - \omega_x z,\ \omega_x y - \omega_y x)$$

Move on to angular momentum $\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times(\vec{\omega}\times\vec{r})$

$$\vec{r}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ \omega_y z-\omega_z y & \omega_z x-\omega_x z & \omega_x y-\omega_y x \end{vmatrix}$$

$$= \left[(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z\right]\hat{i}$$
$$+ \left[-xy\,\omega_x + (x^2+z^2)\omega_y - yz\,\omega_z\right]\hat{j}$$
$$+ \left[-xz\,\omega_x - yz\,\omega_y + (x^2+y^2)\omega_z\right]\hat{k}$$

It's inspiring to rewrite the results in matrix form

$$\boxed{\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} m(y^2+z^2) & -mxy & -mxz \\ -mxy & m(x^2+z^2) & -myz \\ -mxz & -myz & m(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}$$

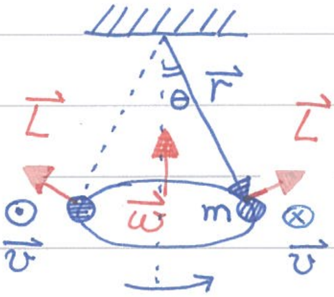
In general, $\vec{L}$ and $\vec{\omega}$ are NOT parallel.

$$\boxed{L_i = \sum_{j=1}^{3} I_{ij}\,\omega_j}$$

Here $I_{ij} = I_{ji}$ is moment of inertia.

It is a rank-2 tensor: $I_{ij} = -m x_i x_j + m r^2 \delta_{ij}$

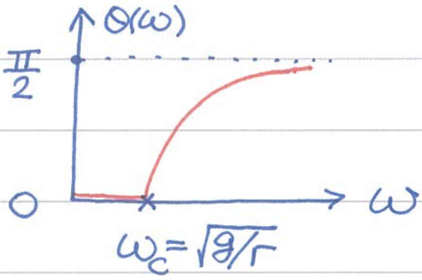
example conical pendulum. Treating the problem as circular motion. $\longrightarrow$ $\cancel{m}g\underbrace{\cancel{\tan\theta}}_{1/\cos\theta} = \cancel{m}(r\cancel{\sin\theta})\,\omega^2$



$\cos\theta = g/r\omega^2 \longrightarrow \boxed{\theta = \cos^{-1}(g/r\omega^2)}$

It is clear that $\vec{L} \nparallel \vec{\omega}$ here

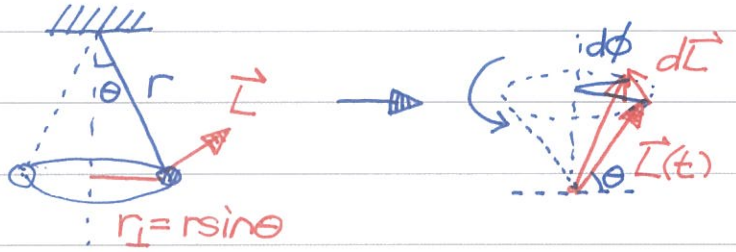
Plot the conical angle $\theta = \theta(\omega)$



$$\boxed{\begin{aligned} &\omega < \omega_c\,, \quad \theta = 0 \\ &\omega > \omega_c\,, \quad \theta = \cos^{-1}(\omega_c^2/\omega^2) \end{aligned}}$$

Need a minimum angular velocity ω_c to maintain the conical pendulum with $\theta \neq 0$!

Now, try to understand the conical pendulum by $\vec{\tau} = \frac{d\vec{L}}{dt}$.



The angular momentum $\vec{L}$ changes with time,

$$d\vec{L} = \vec{L}(t+dt) - \vec{L}(t) \neq 0$$

First of all, the angular momentum is

$$|\vec{L}| = r \cdot p = r \cdot m r_\perp \omega = mr^2\omega\sin\theta\,, \quad \text{where } r_\perp = r\sin\theta$$

$$|d\vec{L}| = (L\cos\theta)\cdot d\phi = L\cos\theta\cdot\omega dt$$

$$\longrightarrow \quad \left|\frac{d\vec{L}}{dt}\right| = L\omega\cos\theta = mr^2\omega^2\cos\theta\sin\theta.$$

On the other hand, the torque caused by gravity is

$$\tau = |\vec{r}\times m\vec{g}| = mgr_\perp = mgr\sin\theta$$

One can check that $\vec{\tau}$ and $d\vec{L}/dt$ point in the same direction and the EOM $\vec{\tau} = d\vec{L}/dt$ reduces to $\tau = \left|\frac{d\vec{L}}{dt}\right|$.

$$\cancel{m}g\cancel{r}\cancel{\sin\theta} = \cancel{m}r^{\cancel{2}}\omega^2\cos\theta\cancel{\sin\theta} \quad \longrightarrow \quad \boxed{\cos\theta = \frac{g}{r\omega^2}}$$

One sees that even rotation around a fixed axis with constant $\vec{\omega}$ can be non-trivial — the angular momentum $\vec{L} = \vec{L}(t)$ is always changing

⑩ Rotational Kinectic energy.

$$K = \frac{1}{2}m(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\,(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})$$

Make use of the identity for vector products,

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

The kinetic energy for a single particle in rotation is

$$K = \frac{1}{2}m\left[(\vec{\omega}\cdot\vec{\omega})(\vec{r}\cdot\vec{r}) - (\vec{\omega}\cdot\vec{r})(\vec{r}\cdot\vec{\omega})\right]$$

$$= \frac{1}{2}m\left[r^2(\omega_x^2+\omega_y^2+\omega_z^2) - (x\omega_x+y\omega_y+z\omega_z)^2\right]$$

$$= \frac{1}{2}m\left[(r^2-x^2)\omega_x^2 + (r^2-y^2)\omega_y^2 + (r^2-z^2)\omega_z^2 - 2xy\,\omega_x\omega_y - 2yz\,\omega_y\omega_z - 2zx\,\omega_z\omega_x\right]$$

$$\rightarrow \boxed{K = \frac{1}{2}(\omega_x, \omega_y, \omega_z)\underbrace{\begin{pmatrix} m(r^2-x^2) & -mxy & -mxz \\ -mxy & m(r^2-y^2) & -myz \\ -mxz & -myz & m(r^2-z^2) \end{pmatrix}}_{I_{ij}}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}}$$

Or, it can be written as $K = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\omega_i I_{ij}\omega_j$

The moment of inertia I_{ij} appears in the expression again. Apparently, I_{ij} plays a central role in rotational dynamics.

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2013.1107