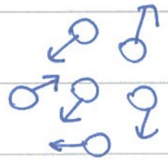
HH0089 Is Energy Always Conserved?

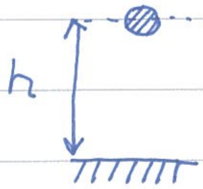
In high-school education, you may have learned different forms of energies already.

$K = \frac{1}{2}mv^2$



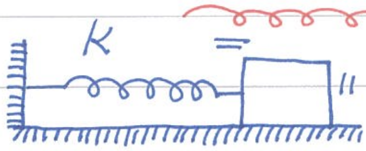
$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$

Kinetic energy for moving particles. — just add up all contributions. In addition, we also know some simple potential energies ...



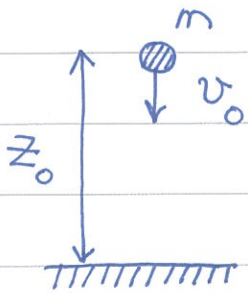
$U_g = mgh$

$U_s = \frac{1}{2}kx^2$



For potential energy, one needs to specify where the ZERO is. For instance, we often choose $U_s = 0$ at the equilibrium point for the spring.

Conservation of energy from EOM.



Write down the EOM first: $m\frac{dv}{dt} = -mg$

$\rightarrow \boxed{\frac{dv}{dt} = -g}$ find the solution $v(t)$

$$\int_0^t \frac{dv}{dt}\,dt = \int_0^t -g\,dt \rightarrow v(t) - v_0 = -gt$$

By integrating EOM, we find the solution for $v(t)$:

$\boxed{v(t) = v_0 - gt}$ note that $v(t) = \frac{dz}{dt}$.

$$\rightarrow \int_0^t \frac{dz}{dt}\,dt = \int_0^t (v_0 - gt)\,dt$$

$$z(t) - z_0 = v_0 t - \frac{1}{2}gt^2 \rightarrow \boxed{z(t) = z_0 + v_0 t - \frac{1}{2}gt^2}$$

An interesting relation emerges if one tries to eliminate the variable t. ☺

$$v = v_0 - gt \longrightarrow t = \frac{1}{g}(v_0 - v)$$

substitute into $z(t)$ solution

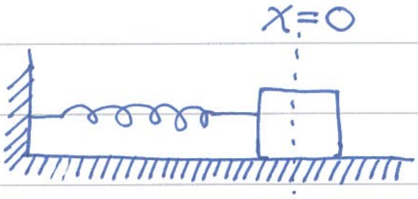
$$z = z_0 + \frac{1}{g} v_0 (v_0 - v) - \frac{1}{2} \cancel{g} \frac{1}{g^{\cancel{2}}} (v_0 - v)^2$$

After some algebra, the above equation can be brought into an inspiring form.

$$z = z_0 + \frac{1}{2g} v_0^2 - \frac{1}{2g} v^2 \longrightarrow \boxed{\frac{1}{2}mv^2 + mgz = \frac{1}{2}mv_0^2 + mgz_0}$$

From the above result, we know EOM tells us the energy is conserved.

Let us turn to the SHM. Suppose $x(0) = A$ and $v(0) = 0$.



We again write down the EOM:

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

We already learned the solution in previous lectures.

$$x(t) = A\cos(\omega_0 t), \quad v(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of SHM.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega_0^2 \sin^2(\omega_0 t) + \frac{1}{2}kA^2\cos^2(\omega_0 t)$$

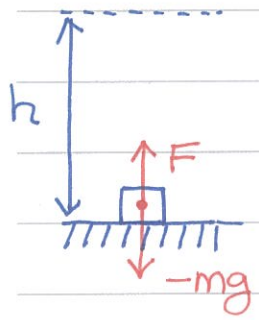
$$= \frac{1}{2}kA^2\left[\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right].$$

Because $\sin^2 z + \cos^2 z = 1$, we obtain the interesting relation for SHM:

$$\boxed{\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{const.}}$$

Ⓘ Work - kinetic energy theorem.

From previous examples, we guess EOM and E-conservation are closely connected. Is this true? What about another quantity "work"? Is it also related to E-conservation?



Suppose we apply an upward force F, equal in magnitude with gravity force -mg. Lift the object from z=0 to z=h.

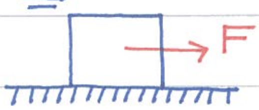
$$W_F = F \cdot h = mgh > 0$$

We often say that the work done by the external force W_F is transformed into the increase of the potential energy. But, the gravitational force also does some work....

$$W_G = (-mg) \cdot h = -mgh, \quad \text{in fact } W_F + W_G = 0 \ !$$

Because the work done by the total external force is ZERO, why should the energy increase?

Short answer is — "work" is related to the change of kinetic energy, not energy.



$$F = \frac{dp}{dt} = m\frac{dv}{dt}$$ integrate over dx on both sides.

$$\int_1^2 F\,dx = \int_1^2 m\frac{dv}{dt}\,dx$$

① Work is defined $W \equiv \int_1^2 F\,dx$

② $\frac{dv}{dt}dx = dv\frac{dx}{dt} = v\,dv$

$$W = \int_1^2 mv\,dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The above result is the so-called work-energy theorem.

$$\boxed{W = \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}$$

Note that the theorem is just an integrated form of EOM.

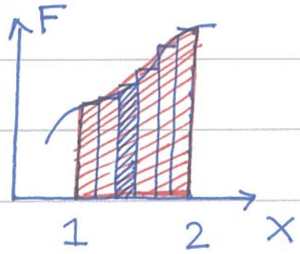
⑩ Definition of work, revisited.

In general, the work is defined as the inner product of the force and the displacement,

$$W = \vec{F}\cdot\Delta\vec{r} = |\vec{F}|\,|\Delta\vec{r}|\cos\theta \quad \leftarrow \text{geometric form}$$

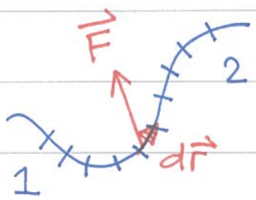
$$= F_x\Delta x + F_y\Delta y + F_z\Delta z \quad \leftarrow \text{component sum.}$$

When the force is not constant, one can add up all contributions by integration.



$$\boxed{W = \int_1^2 F\,dx}$$

The work equals the area in red.

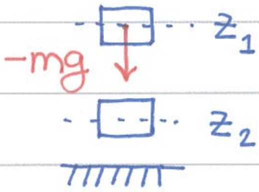


For the general case in 3D, computing "work" requires hard working ☺

$$\boxed{\begin{aligned} W &= \int_1^2 \vec{F}\cdot d\vec{r} \\ &= \int_1^2 F_x\,dx + F_y\,dy + F_z\,dz \end{aligned}}$$

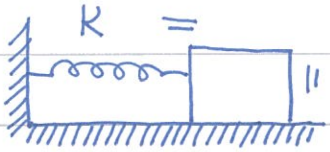
As is clear from the definition, if $\vec{F}\cdot d\vec{r} = 0$, no work is done. The vector property here is of crucial importance.

Here, let's work out some examples



$$W = \int_1^2 (-mg)\,dz = -mg(z_2 - z_1)$$

The work done by gravity looks similar to the gravitational potential energy!



$$W = \int_1^2 (-kx)\,dx = -\frac{1}{2}k(x_2^2 - x_1^2)$$

Again, the work done by the spring force looks similar to the elastic potential energy.

Finally, we rederive the work-energy theorem in 3D.

$$\vec{F} = m\frac{d\vec{v}}{dt} \longrightarrow \int_1^2 \vec{F}\,d\vec{r} = \int_1^2 m\frac{d\vec{v}}{dt}\cdot d\vec{r}$$

Again, $\frac{d\vec{v}}{dt}\cdot d\vec{r} = d\vec{v}\cdot\frac{d\vec{r}}{dt} = \vec{v}\cdot d\vec{v} = v_x dv_x + v_y dv_y + v_z dv_z$

The integrated EOM now takes the form,

$$\boxed{\begin{aligned} W &= \int_1^2 m v_x dv_x + m v_y dv_y + m v_z dv_z \\ &= \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)\Big|_1^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \end{aligned}}$$

The work-energy theorem takes the same form as in 1D.

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