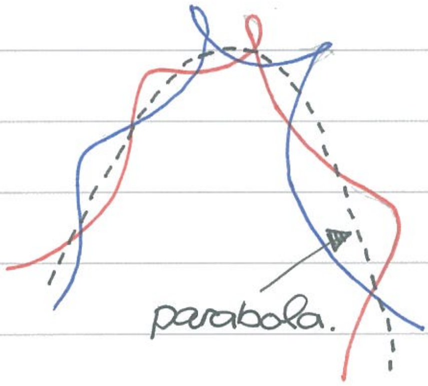
豪豬筆記

HH0087 Center of Mass.

Consider the trajectories of tossed baton 

in the air. They look quite complicated. BUT! If we define a point



$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2), \quad \text{where } M = m_1 + m_2,$$

its trajectory is a simple parabola. This amazing point is the "center of mass".

⊘ Dynamics of CM.

Consider a two-particle system

$$m_1\frac{d^2x_1}{dt^2} = F_1 + f_{12}$$

$$m_2\frac{d^2x_2}{dt^2} = F_2 + f_{21}$$

$$\longrightarrow \quad \frac{d^2}{dt^2}(m_1x_1 + m_2x_2) = F_1 + F_2$$

Define the center of mass coordinate $x_{cm} = \frac{1}{M}(m_1x_1 + m_2x_2)$

$$M\frac{d^2x_{cm}}{dt^2} = F_{ex}$$

The dynamics of CM only depends on the external force!

The idea can be generalized to N-particle system in 3D,

$$\boxed{\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N)}, \quad M = m_1 + m_2 + \cdots + m_N$$

total mass.

Taking time derivatives on both sides,

$$\vec{V}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}(m_1\vec{V}_1 + m_2\vec{V}_2 + \cdots + m_N\vec{V}_N)$$

One more derivative leads to

$$\vec{a}_{cm} = \frac{d^2\vec{r}_{cm}}{dt^2} = \frac{1}{M}(m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_N\vec{a}_N)$$

The EOM for the center of mass is simple,

$$m_1\vec{a}_1 = \vec{F}_1 + (\vec{f}_{12} + \vec{f}_{13} + \cdots + \vec{f}_{1N})$$

$$m_2\vec{a}_2 = \vec{F}_2 + (\vec{f}_{21} + \vec{f}_{23} + \cdots + \vec{f}_{2N})$$

$$\vdots$$

$$+ \quad \underline{m_N\vec{a}_N = \vec{F}_N + (\vec{f}_{N1} + \vec{f}_{N2} + \cdots + \vec{f}_{NN-1})}$$

$$\Rightarrow \quad m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_N\vec{a}_N = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N$$

That is to say, $\boxed{M\vec{a}_{cm} = \vec{F}_{ex}}$. The motion of CM only depends on the total external force.

In the absence of $\vec{F}_{ex}$, $M\vec{a}_{cm} = 0 \Rightarrow M\frac{d\vec{V}_{cm}}{dt} = 0$

$$M\vec{V}_{cm} = \text{const} \Rightarrow \boxed{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_N\vec{v}_N = \text{const}}$$

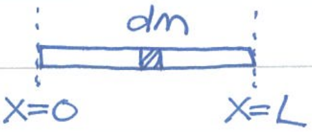
So you can see the conservation of momentum when the external force vanishes.

① Continuous limit When the particle number is huge, it is often possible to define the CM by integrals,

$$\boxed{\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm}$$

We would work out some examples to get familiar with the notation.

example 1: The linear density $\lambda \equiv M/L$, $dm = \lambda dx$

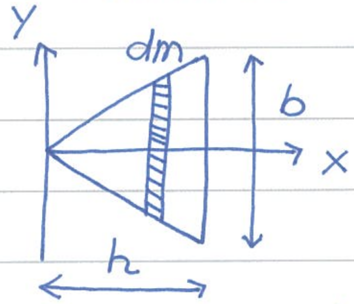


$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L \lambda x\, dx$$

$$= \frac{\lambda}{M}\frac{x^2}{2}\Big|_0^L = \frac{\lambda}{M}\frac{L^2}{2} = \frac{L}{2} \#$$

The CM locates at the center of the rod — quite reasonable. Suppose $\lambda = \lambda(x)$ is NOT uniform now. Do you know how to compute x_{cm}?

example 2 :



$\sigma = M/A = 2M/hb$

$dm = \sigma . dA = \sigma . (\frac{x}{h} . b) . dx$

$$dm = \frac{\sigma b}{h} x \, dx$$

According to the definition of CM,

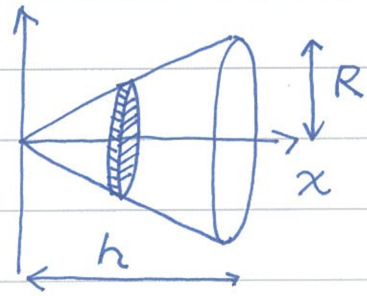
$$x_{cm} = \frac{1}{M} \int x \, dm = \frac{1}{M} \int_0^h x . \frac{\sigma b}{h} x \, dx = \frac{\sigma b}{Mh} \int_0^h x^2 dx$$

$$\rightarrow \quad x_{cm} = \frac{2}{h^2} . \frac{1}{3} h^3 = \frac{2}{3} h \; \#$$

closer to the bottom

example 3 :

$\rho = M/V = M/\frac{1}{3}\pi R^2 h = 3M/\pi R^2 h.$



One needs to find the expression for the infinitesimal element of mass,

$$dm = \rho \, dV = \rho . \pi r^2 dx \qquad r = R . (x/h)$$

$$= \frac{\rho \pi R^2}{h^2} x^2 dx$$

Now we are ready to compute the CM coordinate x_{cm}.

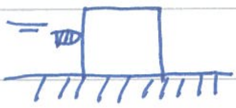
$$x_{cm} = \frac{1}{M} \int x \, dm = \frac{\rho \pi R^2}{M h^2} \int_0^h x^3 dx = \frac{3}{h^3} . \frac{1}{4} h^4$$

$$\rightarrow \quad x_{cm} = \frac{3}{4} h \; \#$$

It is even closer to the bottom

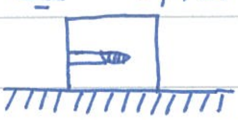
⊘ 1D collision revisited.

before



$V_1 = V_0 , \; V_2 = 0$

after



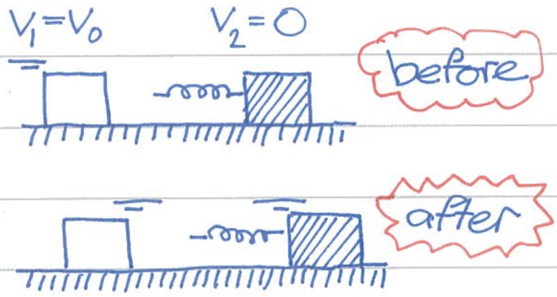
$V_1' = V_2' = \frac{m_1}{m_1 + m_2} V_0$

No external force → The dynamics of the CM is trivial

$$V_{cm} = \frac{dx_{cm}}{dt} = \frac{m_1 V_0}{m_1 + m_2}$$

$$\boxed{x_{cm}(t) = x_0 + \frac{m_1}{m_1 + m_2} V_0 t}$$

CM is a pretty good description.

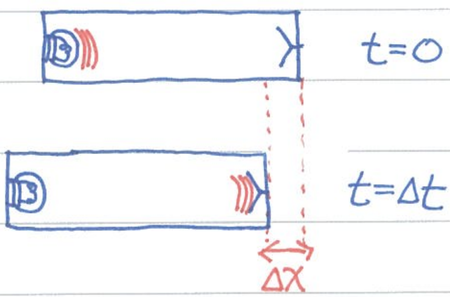


$$V_1'=\frac{m_1-m_2}{m_1+m_2}V_0 \qquad V_2'=\frac{2m_1}{m_1+m_2}V_0$$

Again, no external force, the CM dynamics is trivial, $X_{cm}=X_0+\frac{m_1 V_0}{m_1+m_2}t$.

However, its trajectory does not faithfully capture the motion of the 2-body system.

Ⓓ $E=mc^2$. Here we would like to present an interesting gedanken experiment that leads to the famous $E=mc^2$.



A light flash is generated at $t=0$ and then absorbed at later time Δt. Because the light flash carries both energy E and momentum P (related by $E=PC$ from Maxwell equations), the box will move in opposite direction as shown.

But, here comes the puzzle ∘∘∘ No external force, the CM was at rest, but $\Delta X_{cm}=\Delta x\neq 0$?!

To cure the problem, assume the light flash carries mass m. Because the CM is at rest,

$$\Delta x_{cm}=0 \implies m\Delta x_\ell + M\Delta X_B = 0$$

$$m(L-\Delta x)+M(-\Delta x)=0 \rightarrow \boxed{\frac{\Delta X}{L-\Delta X}=\frac{m}{M}}$$

On the other hand, from the basic kinematics,

$$\Delta x = v\Delta t = \frac{P}{M}\Delta t \quad \text{and} \quad \Delta t=\frac{L-\Delta X}{c}.$$

Therefore, $\Delta x = \frac{p}{M} \cdot \left(\frac{L - \Delta x}{c} \right)$

$$\rightarrow \boxed{\frac{\Delta x}{L - \Delta x} = \frac{p}{Mc}}$$

Note that the box and the light flash carry the same momentum p with opposite directions.

Compare both equations,

$$\frac{m}{M} = \frac{p}{Mc} \rightarrow p = mc$$

Furthermore, making use of the relation $E = pc$ from Maxwell equations, we obtain the relation

$\boxed{E = pc = mc^2}$ for the traveling light flash

With more advanced techniques, one can show that $E = mc^2$ holds true for general cases!

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