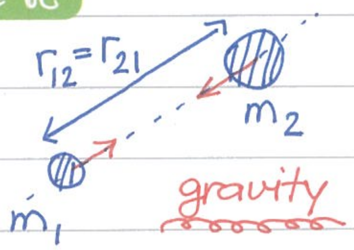
豪豬筆記

HH0086 Do forces always appear in pairs?

Newton's 3rd law – To every action there is always an equal but opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.



$$\vec{f}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \quad (\text{acting on } 1)$$

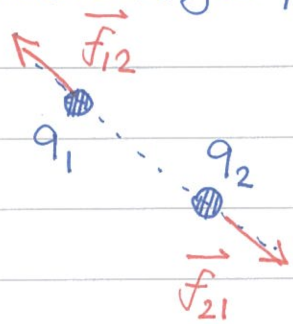
$$\vec{f}_{21} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21} \quad (\text{acting on } 2)$$

Note that $\hat{r}_{12} = -\hat{r}_{21}$ and $r_{12} = r_{21} = |\vec{r}_1 - \vec{r}_2|$ is the mutual distance, therefore,

$$\boxed{\vec{f}_{12} + \vec{f}_{21} = 0}$$

Newton's 3rd law!

Similarly, it also applies to the Coulomb's law between two charged particles,



$$\vec{f}_{12} = -\vec{f}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r_{12}^2}\hat{r}_{12}$$

As will be shown later, Newton's 3rd law ensures the conservation of momentum for the whole system.

ⓘ Momentum-impulse theorem.

Starting from Newton's 2nd law $\vec{F} = \frac{d\vec{p}}{dt}$, integrating over time leads to

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\cdot dt$$

$$\rightarrow \quad \boxed{\Delta\vec{p} \equiv p(t_2) - p(t_1) = \int_{t_1}^{t_2} \vec{F}\,dt}$$

For convenience, introduce a vector called "impulse",

$$\underbrace{\vec{J} \equiv \int_{t_1}^{t_2} \vec{F}\, dt}_{} \Rightarrow \boxed{\Delta\vec{p} = \vec{J}}$$ momentum-impulse Thm.

If $\vec{J}=0$, the momentum of the particle won't change.

Now try to generalize the theorem to 2-particle system. The equations of motion are

$$\boxed{\begin{aligned} \frac{d\vec{p}_1}{dt} &= \vec{F}_1 + \vec{f}_{12} \\ \frac{d\vec{p}_2}{dt} &= \vec{F}_2 + \vec{f}_{21} \end{aligned}} \longrightarrow \begin{aligned} \Delta\vec{p}_1 &= \vec{J}_1 + \vec{\jmath}_{12} \\ \Delta\vec{p}_2 &= \vec{J}_2 + \vec{\jmath}_{21} \end{aligned}$$

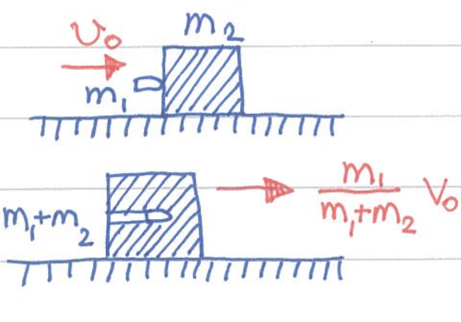
Here $\vec{F}_1, \vec{F}_2$ are external forces acting on 1 and 2, while $\vec{f}_{12}, \vec{f}_{21}$ are internal forces due to mutual interactions.

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = \underbrace{\vec{J}_1 + \vec{J}_2}_{\text{from } \vec{F}_{ex}!} + \underbrace{\vec{\jmath}_{12} + \vec{\jmath}_{21}}_{\text{zero!}}, \qquad \vec{J}_{ex} \equiv \int_{t_1}^{t_2} \vec{F}_{ex}\, dt = \vec{J}_1 + \vec{J}_2$$

Finally, we find the momentum-impulse theorem for the 2-particle system, $\boxed{\Delta\vec{P} = \vec{J}_{ex}}$ where $\vec{P} = \vec{p}_1 + \vec{p}_2$

In the absence of external forces, $\vec{J}_{ex}$ vanishes and thus $\Delta\vec{P} = 0$. That is to say, the total momentum of the system is conserved when the external force vanishes.

Returning to the previous example.



Before collision:

$$p_1 + p_2 = m_1 v_0 + m_2 0 = \underbrace{m_1 v_0}_{}$$

The same!

After collision:

$$p_1' + p_2' = (m_1+m_2) \cdot \frac{m_1 v_0}{m_1+m_2} = \underbrace{m_1 v_0}_{}$$

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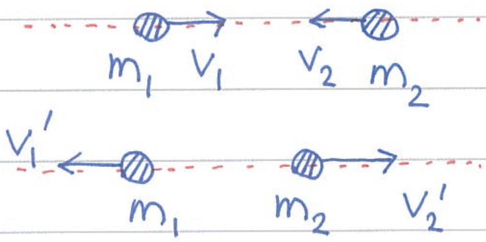
⦰ Collision in 1D



Let's consider elastic collision

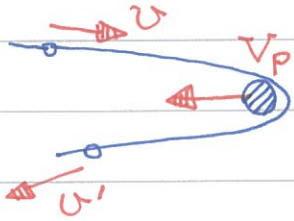
$$\boxed{\begin{gathered} m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \\ \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 \end{gathered}}$$

From the conservation of momentum and kinetic energy, one can find the velocities after collision,

$$\boxed{\begin{aligned} v_1' &= \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \\ v_2' &= \frac{2m_1}{m_1 + m_2} v_1 + \frac{m_2 - m_1}{m_1 + m_2} v_2 \end{aligned}}$$

Notice the symmetric expression in the formula.

example. Sling-shot effect



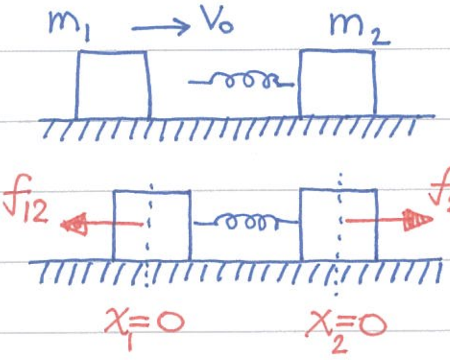
Because the mass of the planet is huge,

$$v' = \frac{m_1 - m_2}{m_1 + m_2} v + \frac{2m_2}{m_1 + m_2} V_p$$

$$\approx -v + 2V_p = -\underbrace{(v + 2|V_p|)}$$

The final speed is larger — stealing some kinetic energy from the planet.

⦰ The power of EOM



Find the action and reaction first,

$$\underbrace{f_{12} = -f_{21} = -k(x_1 - x_2)}$$

Write down the EOM:

$$\boxed{\begin{aligned} m_1 \frac{d^2 x_1}{dt^2} &= -k(x_1 - x_2) \\ m_2 \frac{d^2 x_2}{dt^2} &= k(x_1 - x_2) \end{aligned}}$$

Newton's 3rd law:

$$f_{12} + f_{21} = 0.$$

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Adding two EOM together

$$m_1 \frac{dv_1}{dt} + m_2 \frac{dv_2}{dt} = -k(x_1 - x_2) + k(x_1 - x_2) = 0$$

$$\rightarrow \boxed{m_1 v_1(t) + m_2 v_2(t) = m_1 v_0} \rightarrow \text{constant in time.}$$

Introduce the relative displacement $x \equiv x_1 - x_2$

$$\frac{d^2x}{dt^2} = -\frac{k}{m_1}(x_1 - x_2) - \frac{k}{m_2}(x_1 - x_2) = -\frac{k}{\mu} x \quad , \mu = \frac{m_1 m_2}{m_1 + m_2}$$

reduced mass.

Rewriting the above equation into standard SHM form,

$$\boxed{\frac{d^2x}{dt^2} + \left(\frac{k}{\mu}\right) x = 0} \rightarrow \omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

Initial conditions: $x(t=0) = 0$ and $V(t=0) = V_1(0) - V_2(0) = V_0$

Solution:

$$\boxed{\begin{aligned} x(t) &= \frac{V_0}{\omega_0} \sin(\omega_0 t) \\ V(t) &= V_0 \cos(\omega_0 t) \end{aligned}}$$

$V(t) = V_1(t) - V_2(t)$.

Collecting the solutions together

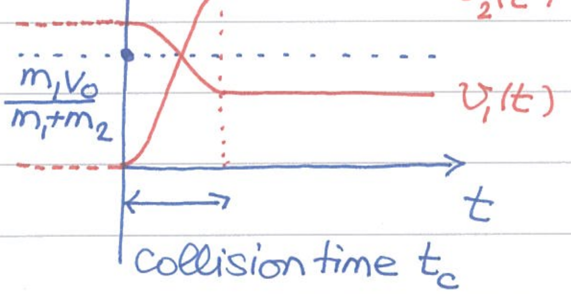
$$m_1 v_1(t) + m_2 v_2(t) = m_1 v_0$$

$$v_1(t) - v_2(t) = v_0 \cos(\omega_0 t)$$

two equations, two unknowns, we can solve for $v_i(t)$.

$$\boxed{\begin{aligned} v_1(t) &= \frac{m_1}{m_1 + m_2} V_0 + \frac{m_2}{m_1 + m_2} V_0 \cos(\omega_0 t) \\ v_2(t) &= \frac{m_1}{m_1 + m_2} V_0 - \frac{m_1}{m_1 + m_2} V_0 \cos(\omega_0 t) \end{aligned}}$$

Q: When does the "collision" end?



At $t = \frac{\pi}{\omega_0}$, the collision ends and the velocities remain constant afterward.

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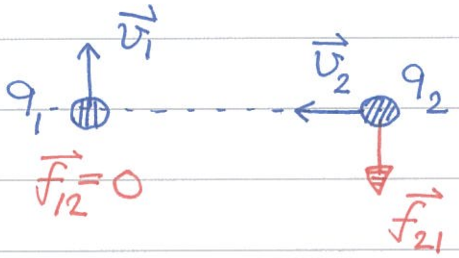
$$v_1' = \frac{m_1 v_0}{m_1+m_2} + \frac{m_2 v_0}{m_1+m_2}\cos\left(\omega_0 \cdot \frac{\pi}{\omega_0}\right) = \frac{m_1-m_2}{m_1+m_2} v_0$$

$$v_2' = \frac{m_1 v_0}{m_1+m_2} - \frac{m_1 v_0}{m_1+m_2}\cos\left(\omega_0 \cdot \frac{\pi}{\omega_0}\right) = \frac{2m_1}{m_1+m_2} v_0$$

The final velocities are the same as predicted in the formula for 1D elastic collision. Note that EOM not only give the final velocities but also all dynamical details during the collision.

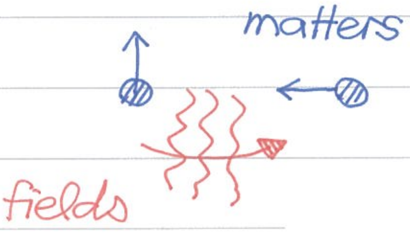
⊘ Breakdown of Newton's 3rd law?

Consider the magnetic forces between two moving charges,



$$\boxed{\begin{aligned} \vec{f}_{12} &= \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r_{12}^2}\,\vec{v}_1 \times (\vec{v}_2 \times \hat{r}_{12}) \\ \vec{f}_{21} &= \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r_{12}^2}\,\vec{v}_2 \times (\vec{v}_1 \times \hat{r}_{21}) \end{aligned}}$$

Because the forces depend on the vector products, they do not add up to zero, i.e. $\vec{f}_{12} + \vec{f}_{21} \neq 0$ and Newton's 3rd law is no longer valid?! But momentum conservation is still working. Hum, how? ☺



The fields also carry momentum and thus we need to include BOTH matters and fields!

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