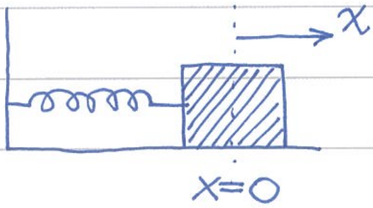
HH0085 Simple Harmonic Motion

Consider a mass attached to a spring. The restoring force follows Hooke's law:



$$\boxed{F = -kx}$$

minus sign is important!

According to Newton's second law, the dynamics is captured by the following differential equation,

$$F = ma \longrightarrow -kx = m\frac{d^2x}{dt^2}$$

introduce $\omega_0 = \sqrt{\frac{k}{m}}$

The above EOM can be simplified,

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$$

$x = C_1 \cos\omega_0 t$, $x = C_2 \sin\omega_0 t$ both satisfy the differential equation. The general solution is just their linear combination,

$$\boxed{x(t) = C_1\cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A\cos(\omega_0 t + \phi)}$$

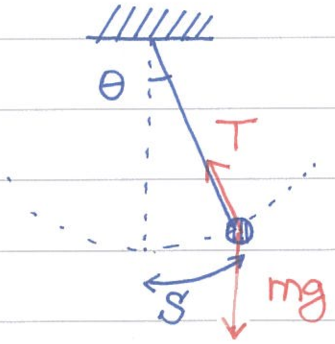
<u>check</u>. velocity $v(t) = \frac{dx}{dt} = -A\sin(\omega_0 t + \phi)\cdot\omega_0$

acceleration $a(t) = \frac{dv}{dt} = -A\omega_0 \cos(\omega_0 t + \phi)\cdot \omega_0$

$$\longrightarrow \frac{d^2x}{dt^2} = a(t) = -A\omega_0^2\cos(\omega_0 t + \phi) = -\omega_0^2 x(t). \checkmark$$

Simple pendulum:

The forces can be decomposed in tangent and normal directions.



tangent force: $-mg\sin\theta$
normal force: $T - mg\cos\theta$

Write down the equations of motion:

tangent direction — $F_t = ma_t$

$$\boxed{-mg\sin\theta = m\frac{d^2s}{dt^2} = ml\frac{d^2\theta}{dt^2}} \leftarrow S(t) = l\,\theta(t)$$

normal direction — $F_n = mv^2/R$

$$\boxed{T - mg\cos\theta = m\left(\frac{ds}{dt}\right)^2 \cdot \frac{1}{l} = ml\left(\frac{d\theta}{dt}\right)^2}$$

Simplify the EOM in the tangent direction,

$$\boxed{\frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\sin\theta = 0} \quad \text{introduce } \omega_0 = \sqrt{\frac{g}{l}} \text{ natural frequency.}$$

In principle, we can find the dynamics of the angle, $\theta = \theta(t)$. The EOM in the normal direction can be used to solve for the varying tension $T = T(t)$ once the motion of θ is known.

For small oscillatory angle, $\sin\theta \approx \theta$. The EOM simplifies,

$$\boxed{\frac{d^2\theta}{dt^2} + \omega_0^2\theta \approx 0} \leftarrow \text{exactly the same as S.H.M.}$$

Therefore, the general solution is $\theta(t) = A\cos(\omega_0 t + \phi)$

Some simple kinematics for SHM. The period T is defined as the minimum time such that $\theta(t) = \theta(t+T)$

$$A\cos(\omega_0 t + \phi) = A\cos(\omega_0 t + \omega_0 T + \phi) \rightarrow \boxed{\omega_0 T = 2\pi}$$

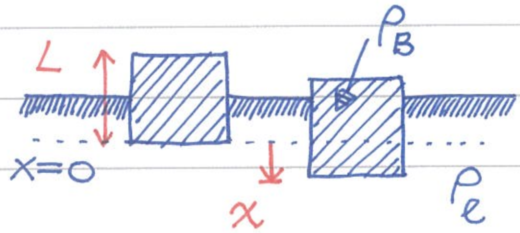
Thus, the period is $\quad T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{l}{g}} = 2\pi\sqrt{\dfrac{m}{k}}$

important!

The frequency is defined as

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{\omega_0}{2\pi}$$

Floating block. The floating force is proportional to the volume below the level. It is convenient to set $x=0$ where $\vec{F}_b + \vec{F}_g = 0$.

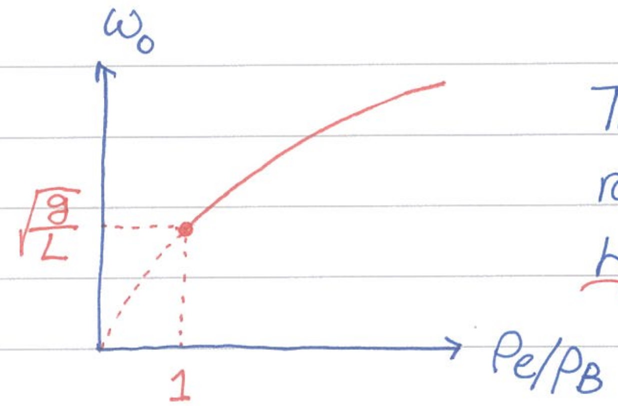


$$\rightarrow \quad \vec{F}_b + \vec{F}_g = -(\rho_e A x) g = -(A\rho_e g) x \propto x$$

Write down the EOM for the floating block.

$$-(\cancel{A}\rho_e g) x = (\rho_B \cancel{A} L) \frac{d^2x}{dt^2} \quad \rightarrow \quad \boxed{\frac{d^2x}{dt^2} + \left(\frac{\rho_e g}{\rho_B L}\right) x = 0}$$

Its dynamics is simple harmonic oscillation ☺ The natural frequency is $\omega_0^2 = \frac{\rho_e g}{\rho_B L} \rightarrow \boxed{\omega_0 = \sqrt{\frac{\rho_e g}{\rho_B L}}}$



The frequency grows as the density ratio ρ_e/ρ_B becomes larger. What happens when $\rho_e/\rho_B \le 1$?

Conserving quantity. For SHM, the trajectory is described by $x(t) = A\cos(\omega_0 t + \phi)$. Because A is the maximum displacement and thus corresponds to the amplitude.

$$v(t) = \frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi). \quad \text{where } \omega_0^2 = \frac{k}{m}.$$

$$\rightarrow \quad \omega_0^2 x^2 + v^2 = A^2\omega_0^2 \left[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)\right] = A^2\omega_0^2$$

i.e. $\frac{k}{m} x^2 + v^2 = \text{Const}$ does not change with time.

The usual convention is to write the "conservation law" in slightly different form.

$$\boxed{\underbrace{\frac{1}{2}kx^2}_{\text{potential energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{kinetic energy.}} = \underbrace{E}_{\text{total energy of the system}}}$$

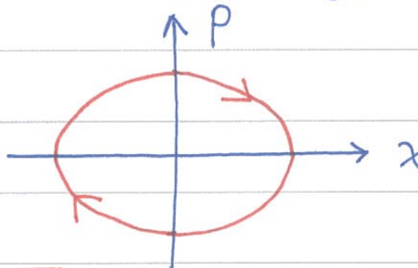
Although the conservation of energy is <u>NOT</u> as powerful as Newton's EOM, it remains valid even when Newtonian mechanics fails ☺

⊘ SHM in phase space

Phase space ≡ x-p space.

For N particles moving in 3D space, the corresponding phase space is 6N dimensional.

$$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \rightarrow \frac{p^2}{2mE} + \frac{x^2}{2E/k} = 1$$



The trajectory in the phase space is an ellipse!

$$\boxed{\left(\frac{p}{p_0}\right)^2 + \left(\frac{x}{x_0}\right)^2 = 1}$$

$p_0 = \sqrt{2mE}$
$x_0 = \sqrt{2E/k}$

Note that the trajectory in the phase space is similar to that in Lotka-Volterra model for predator-prey ecosystem. Why so? We would show later that SHM is a generic result for stable equilibrium under perturbations.

In classical mechanics, one can change the energy continuously. However, in quantum world, it is observed in experiments that the area enclosed by the trajectory in phase space is quantized,

$$\boxed{\oint p\,dx = (n+\tfrac{1}{2})\,h}$$

h — Planck constant

$n = 0, 1, 2, 3, \cdots$ quantized.

From basic calculus,

$$\oint p\,dx = \text{area of ellipse} = \pi \underbrace{p_0}_{\sqrt{2mE}} \underbrace{x_0}_{\sqrt{2E/k}}$$

$$= \frac{2\pi}{\omega_0} E$$

The quantization rule implies that the energy is quantized as below

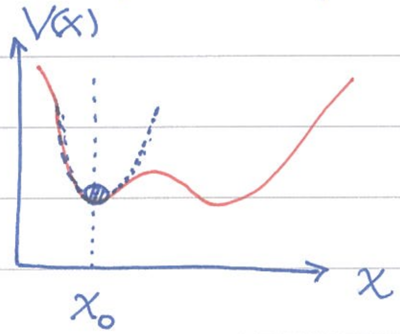
$$\frac{2\pi}{\omega_0} E = \left(n+\frac{1}{2}\right) h \longrightarrow \boxed{E = \left(n+\frac{1}{2}\right)\hbar\omega_0} \qquad \hbar \equiv \frac{h}{2\pi}$$

The energy quantization is closely related to Einstein's assumption for photons $E = nhf = n\hbar\omega = 0, \hbar\omega, 2\hbar\omega, \cdots\cdots$

The $\frac{1}{2}\hbar\omega_0$ is the so-called zero-point energy – important but often can be ignored in experiments.

⊘ Why SHM is so common in nature?

For simplicity, consider 1D motion under the influence of some potential profile. Near the stable equilibrium,



$$\boxed{V(x) \approx V(x_0) + \frac{1}{2}k(x-x_0)^2}$$

i.e. the force satisfies Hooke's law $F = -k(x-x_0)$ where x_0 is the stable equilibrium point.

Therefore, as long as $(x-x_0)$ is small, the dynamics is roughly $x - x_0 = A\cos(\omega_0 t + \phi)$ — SHM is everywhere!

清大東院

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