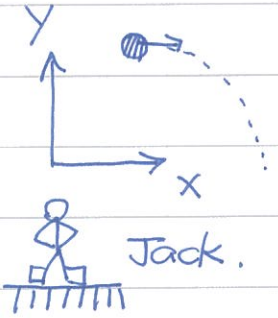
HH0084 Newton's second law

The trajectory of a particle satisfies Newton's second law.
$$\boxed{\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}}$$
differential equation

As long as we know how to solve the differential equation, we can obtain the trajectory.

For instance, the trajectory of a falling body satisfies the equations of motion:



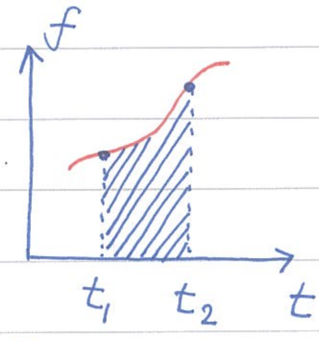
$$\boxed{\begin{aligned} \frac{d^2x}{dt^2} &= 0 \\ \frac{d^2y}{dt^2} &= -g \end{aligned}}$$

$$x(t) = a_0 + a_1 t$$
$$y(t) = b_0 + b_1 t - \frac{1}{2}gt^2$$

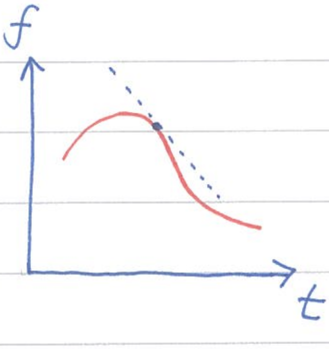
What are the meanings of these constants a_0, a_1, b_0, b_1?

From this simple example, we know calculus is important.

⓪ Some basic calculus.



$$\text{shaded area} = \lim_{\Delta t \to 0} \sum_i f(t_i)\Delta t$$
$$\equiv \int_{t_1}^{t_2} f(t)\, dt$$



$$\text{Slope} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \equiv \frac{df}{dt}.$$

The beauty of calculus is that "integration" and "differentiation" are related ☺

The fundamental theorem of calculus.

$$\boxed{\int_{t_1}^{t_2} f(t)\,dt = F(t_2) - F(t_1), \quad \text{where } \frac{dF}{dt} = f}$$

Example: $f(t) = t^2 \longrightarrow F(t) = \frac{1}{3}t^3 + \text{Const}$

$$\int_0^1 t^2\,dt = \frac{1}{3}t^3 \Big|_0^1$$

$$= \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3} \#$$

Another useful tool is Taylor expansion ☺

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots\cdots$$

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots\cdots$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \cdots\cdots$$

Knowing these series helps us to find the solution of differential equations. Some interesting properties you can derive ....

$$\frac{de^z}{dz} = 0 + 1 + \frac{1}{2!}(2z) + \frac{1}{3!}(3z^2) + \frac{1}{4!}(4z^3) + \cdots\cdots$$

$$= 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots\cdots = e^z$$

The differentiation of e^z is again e^z!

$$\frac{d(\cos z)}{dz} = 0 - \frac{1}{2!}(2z) + \frac{1}{4!}(4z^3) - \frac{1}{6!}(6z^5) + \cdots\cdots$$

$$= -\left[z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots\cdots\right] = -\sin z$$

Thus, $(\cos z)' = -\sin z$. Similarly $(\sin z)' = \cos z$

⦰ **Falling rain.** A rain drop of mass m experiences two forces — gravity and friction.

$$\boxed{f = -bv}$$

Now, write down the equation of motion for the raindrop. ☺

$$\boxed{m\frac{dv}{dt} = mg - bv}$$

The physics part is done and we only need to solve the differential equation.

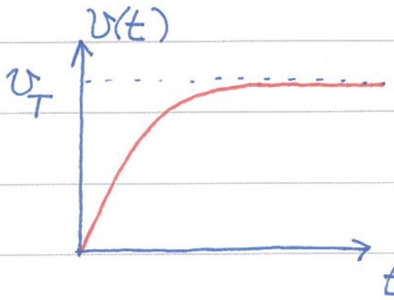
Choose $u(t) \equiv v(t) - v_T$, where the terminal velocity $v_T = mg/b$. The EOM simplifies $\frac{du}{dt} = \left(-\frac{b}{m}\right)u$

Solving for $u(t)$:

$$u(t) = u(0)\, e^{-(b/m)t} \longrightarrow v(t) = v_T + u(t)$$
$$= v_T + u(0)\, e^{-(b/m)t}$$

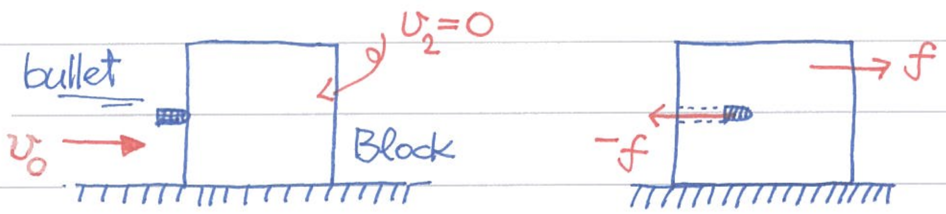
Work out the initial condition, $u(0) = v(0) - v_T$, the final soln is

$$\boxed{v(t) = v_T + [v(0) - v_T]\, e^{-(b/m)t}} \qquad v(t\to\infty) = v_T$$



For simplicity, choose $v(0)=0$, the rain drop accelerates at the begining but gradually approaches the terminal velocity v_T!

⦰ **Shooting bullet.** Let us consider a slightly more complicated system as below. The friction actually speeds up the Block.



Let us write down the EOMs.

$$m_1 \frac{dv_1}{dt} = -f, \qquad m_2 \frac{dv_2}{dt} = f$$

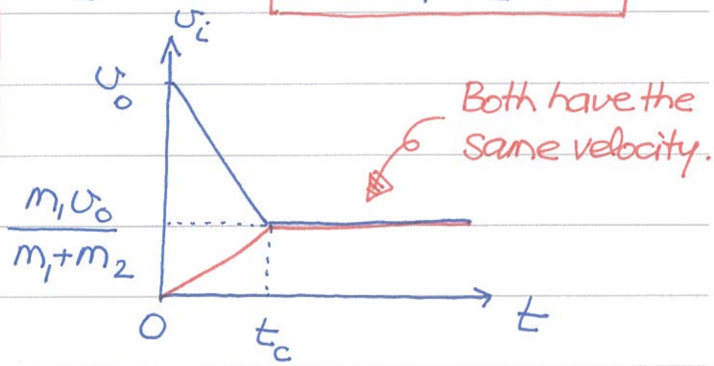
It is rather straightforward to solve the equations,

$$\boxed{\begin{aligned} v_1(t) &= v_0 - \frac{f}{m_1} t \\ v_2(t) &= \frac{f}{m_2} t \end{aligned}}$$

But, this is NOT the whole story. At $t = t_c$, both have the same velocity and the friction disappears.

$$v_1(t_c) = v_2(t_c) \longrightarrow v_0 - \frac{f}{m_1} t_c = \frac{f}{m_2} t_c \qquad \boxed{t_c = \frac{m_1 m_2}{m_1 + m_2} \frac{v_0}{f}}$$

Thus, $\boxed{v_1(t_c) = v_2(t_c) = \dfrac{m_1 v_0}{m_1 + m_2}}$



It is quite easy to check that the momentum is conserved before and after the inelastic "collision".

⊘ 2^{nd} law and momentum.

Newton's 2^{nd} law was originally written as

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

For fixed mass m, it becomes $\vec{F} = m\vec{a}$.

There are dynamical systems better described by the momentum-derivative formalism. Consider a system of variable mass,

Let's compute the momentum change.

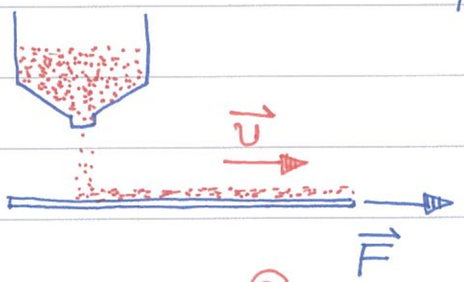
$$d\vec{P} = (M+dM)(\vec{v}+d\vec{v}) - M\vec{v} - dM\vec{u}$$

$$= M\,d\vec{v} + \underbrace{(\vec{v}-\vec{u})\,dM}_{}$$ coming from variable mass.

According to Newton's second law,

$$\boxed{\vec{F} = \frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + (\vec{v}-\vec{u})\frac{dM}{dt}}$$

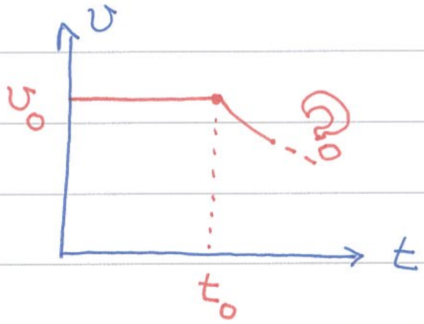
Example: Conveyor belt. One needs to provide an external force F to keep the belt moving.



$$F = M\cancelto{0}{\frac{dv}{dt}} + (v-0)\frac{dM}{dt}$$

$$\rightarrow \boxed{F = v\frac{dM}{dt}}$$ simple

Question Suppose we stop providing the force at $t=t_0$. The mass and velocity at this instant are M_0 and v_0. Do you know how to solve $v(t)$ for later time ?



Try to find the equation of motion first. Then, find the solution $v(t)$.

清大東院
2013.0924