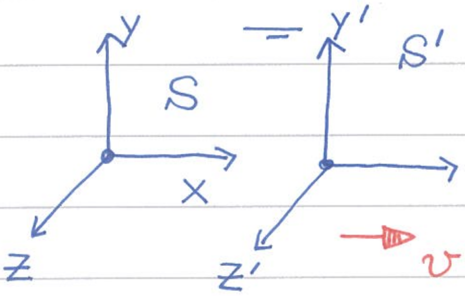
HH0082 Lorentz transformation

Newtonian mechanics is invariant under Galilean transformation,



$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}}$$

absolute time.

Take the time derivative on both sides,

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \longrightarrow u'_x = u_x - v \quad \text{and also} \quad u'_y = u_y, \quad u'_z = u_z$$

This is the velocity addition rule — our common intuitions.

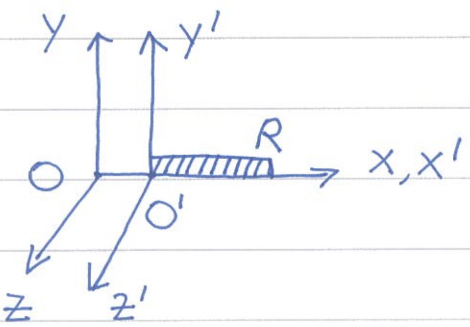
Taking one more derivative gives

$$\frac{du'_x}{dt} = \frac{du_x}{dt},$$ similar identities for y, z directions.

In vector form, the result is very simple $\boxed{\vec{a}' = \vec{a},}$ i.e. the acceleration is invariant under Galilean transformation.

Ⓘ Derivation of Lorentz transformation.

Let us ask Jack and Jill to help us again. A ruler of natural length l_0 is moving with Jill. The origins of the inertia frames are chosen to be



$$\boxed{(x, t) = (0, 0) = (x', t')}$$

Jill's view The ruler is at rest and the trajectory of the end point R is

$$x' = l_0 \quad \text{for all times } t'$$

Jack's view The length of the ruler is contracted to $\ell = \ell_0\sqrt{1-v^2/c^2}$ and the point R is moving at velocity v. Its trajectory is

$$x = \ell + vt = \ell_0\sqrt{1-v^2/c^2} + vt$$

Thus, we find the relation

$$x = x'\sqrt{1-v^2/c^2} + vt \quad \text{OR} \quad \boxed{x' = \gamma(x - vt)}$$

Note that the principle of relativity requires that the form of the transformation from S to S′ be identical to that from S′ to S (except flipping v to $-v$). Thus, we expect

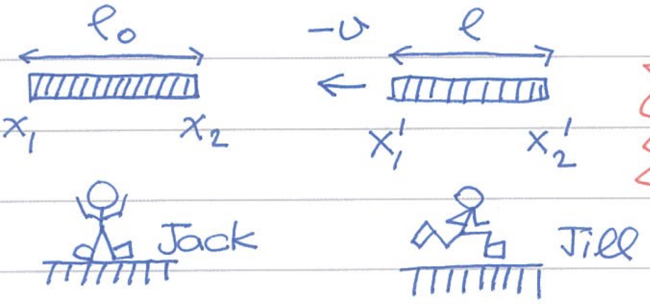
$$\boxed{x = \gamma(x' + vt')} \xrightarrow[\text{eliminate } x']{} \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)}$$

Furthermore, because transverse length is invariant, $y' = y$ and $z' = z$. Collecting all relations together, we find the Lorentz transformation between (x, y, z, t) and (x', y', z', t').

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}} \quad \text{OR} \quad \boxed{\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}}$$

The Lorentz transformation helps us to map events from S to S′, or vice versa. For instance, we can understand length contraction and time dilation without "smart" arguments. It is important to mention that Maxwell equations are invariant under Lorentz transformation.

length contraction revisited ☺



measured at the same time in Jill's frame, $t_1' = t_2'$

$$\ell_0 = x_2 - x_1 = \gamma(x_2' + vt_2') - \gamma(x_1' + vt_1') = \gamma(x_2' - x_1') = \gamma\ell$$

Thus, $\ell = \ell_0\sqrt{1 - v^2/c^2}$ as derived before by light clock.

time dilation revisited

two events @ the same place.

(x_1, t_1) $\downarrow$ (x_2, t_2) $\quad x_1 = x_2$ and $\Delta t_0 = t_2 - t_1$ $\longrightarrow$

(x_1', t_1'), (x_2', t_2') $\quad$ $x_1' \neq x_2'$ anymore

The time interval in Jill's view is

$$t_2' - t_1' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \overset{x_2 = x_1}{=} \gamma(t_2 - t_1) = \gamma\Delta t_0$$

Thus, the time interval is dilated in the moving frame.

⑪ Velocity addition rule:

Let us make use of Lorentz transformation to see how velocities are related.

$$dx' = \gamma(dx - v\,dt)$$
$$dt' = \gamma(dt - v/c^2\,dx)$$
$$\longrightarrow u_x' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

Thus, $u_x' = \dfrac{dx/dt - v}{1 - \frac{v}{c^2}(dx/dt)} \longrightarrow \boxed{u_x' = \dfrac{u_x - v}{1 - vu_x/c^2}}$

Special relativity gives rise to a non-trivial factor $(1 - vu_x/c^2)^{-1}$. You will see how this factor makes the speed of light constant later.

Similarly, $dy' = dy$, $dz' = dz$.

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{u_y}{\gamma(1 - vu_x/c^2)}$$

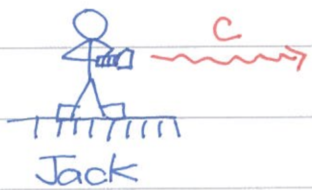
The same transformation for u_z'.

velocity addition rule

$$\boxed{\begin{aligned} u_x' &= \frac{u_x - v}{1 - vu_x/c^2} \\ u_y' &= \frac{u_y}{\gamma(1 - vu_x/c^2)} \\ u_z' &= \frac{u_z}{\gamma(1 - vu_x/c^2)} \end{aligned}}$$

Note that $u_y' \leftrightarrow u_y$ and $u_z' \leftrightarrow u_z$ are affected by the velocity component u_x — motion in x, y, z directions are not independent.

speed of light revisited. Consider a light propagating along the x axis. $\vec{u} = (u_x, u_y, u_z) = (c, 0, 0)$

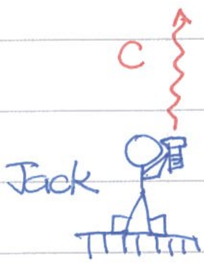


$$u_x' = \frac{c - v}{1 - v/c} = c \;, \quad u_y' = 0 = u_z'$$

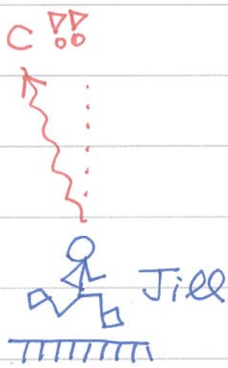
Thus, Jill will observe the same speed c.

What about propagating along the y axis?

$$\vec{u} = (0, c, 0)$$



$$u_x' = -v \;, \quad u_y' = \frac{c}{\gamma} \;, \quad u_z' = 0$$



Thus, the velocity is $\vec{u}' = (-v, \frac{c}{\gamma}, 0)$

The corresponding speed of light is

$$c' = \sqrt{u_x'^2 + u_y'^2 + u_z'^2} = \sqrt{v^2 + c^2(1 - v^2/c^2)} = c$$

One can see that the speed of light is invariant under Lorentz transformation. Very nice

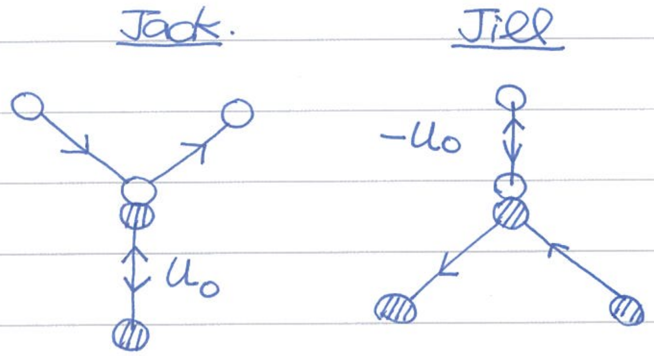
⊘ Relativistic momentum.

In special relativity, we can show that the quantity

$$\boxed{\vec{p} = \frac{1}{\sqrt{1-u^2/c^2}}\, m\vec{u}}$$

is conserved in the elastic collision and thus should be the proper definition of momentum.

Consider the twins Jack and Jill throwing balls at each other to make a symmetric elastic collision as below.



⊘ : Jack
○ : Jill
the same mass.

Both of them throw out the balls vertically at velocity
$(0, u_0, 0)$ Jack.
$(0, -u_0, 0)$ Jill.

In Jill's view, the velocity of Jack's ball is

$$(0, u_0, 0) \longrightarrow \left(-v, \sqrt{1-v^2/c^2}\, u_0, 0\right)$$

Assume the momentum acquires a correction factor that depends on the speed, $\vec{p} = \Gamma(u)\, m\vec{u}$

BEFORE collision.

$$P_y = -\Gamma(u_0)\cdot m u_0 + \Gamma(u_{Jack})\, m\sqrt{1-v^2/c^2}\, u_0$$

where the speed $u_{Jack} = \sqrt{v^2 + u_0^2(1-v^2/c^2)}$.

AFTER collision

$$P_y' = \Gamma(u_0)\, m u_0 - \Gamma(u_{Jack})\, m\sqrt{1-v^2/c^2}\, u_0$$

Momentum conservation requires $P_y = P_y'$ for the elastic collision considered here.

$$\rightarrow \Gamma(u_0)\, m\cancel{u_0} = \Gamma(u_{Jack})\sqrt{1-v^2/c^2}\; m\cancel{u_0}$$

Now take $u_0 \to 0$ limit. The speed $u_{Jack} \to v$

$$\Gamma(0) = \Gamma(v)\sqrt{1-v^2/c^2}\,, \quad \text{but we expect } \Gamma(0)=1$$

Thus, the prefactor is

$$\boxed{\Gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma}$$

The relativistic momentum takes the form:

$$\boxed{\vec{p} = \Gamma(u)\, m\vec{u} = \frac{1}{\sqrt{1-u^2/c^2}}\, m\vec{u}}$$

The correction becomes significant when the speed u is fast.

Newton's 2nd law in its most general form is

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right)}$$

valid relativistically.

Some people like to introduce a relativistic mass

$$m_{rel} = \frac{m}{\sqrt{1-u^2/c^2}} \rightarrow \infty, \text{ as } u \text{ approaches } c.$$

But it can cause confusions from time to time so I won't use the notion in my lectures.

清大東院
2013.0921