EE 2030 Linear Algebra | Chi-chao Chao
Spring 2012

Homework Assignment No. 3
Due 10:10am, April 20, 2012

`Reading:` Strang, Chapter 4, Section 8.5.

`Problems for Solution:`

1. A fundamental theorem in linear algebra is called *Fredholm's Alternative*:

   Exactly one of the following systems has a solution:
   (1) $\boldsymbol{Ax} = \boldsymbol{b}$ (2) $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ with $\boldsymbol{y}^T\boldsymbol{b} \neq 0$.

   Show that it is contradictory for (1) and (2) both to have solutions.

2. (a) Find the orthogonal complement of the plane spanned by the vectors $(1, 1, 2)$ and $(1, 2, 3)$. (*Hint:* Take these to be the rows of $\boldsymbol{A}$ and solve $\boldsymbol{Ax} = \boldsymbol{0}$.)

   (b) Construct a homogeneous equation in three unknowns whose solutions are the linear combinations of the vectors $(1, 1, 2)$ and $(1, 2, 3)$.

3. Consider
$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

   (a) Find the projection matrix $\boldsymbol{P}$ onto the column space of $\boldsymbol{A}$.

   (b) Given $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$, split it into $\boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{x}_{ln}$, where $\boldsymbol{x}_c$ is in the column space of $\boldsymbol{A}$ and $\boldsymbol{x}_{ln}$ is in the left nullspace of $\boldsymbol{A}$.

4. We have four data points with measurements $b = 2, 0, -3, -5$ at times $t = -1, 0, 1, 2$.

   (a) Suppose we want to fit the four data points with a straight line: $b = C_1 + D_1 t$. Find the best least squares straight line fit.

   (b) Suppose we want to fit the four data points with a parabola: $b = C_2 + D_2 t + E_2 t^2$. Find the best least squares parabola fit.

5. Suppose $\boldsymbol{u}$ is an n by 1 unit vector and $\boldsymbol{I}$ is the n by n identity matrix. Consider the matrix $\boldsymbol{Q} = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T$.

   (a) Show that $\boldsymbol{Q}$ is an orthogonal matrix. (It is a reflection, also known as a *Householder transformation.*)

   (b) Show that $\boldsymbol{Q}^2 = \boldsymbol{I}$.

(c) Find $\boldsymbol{Q}_1$ from $\boldsymbol{u}_1^T = (0,1)$ and $\boldsymbol{Q}_2$ from $\boldsymbol{u}_2^T = (0, \sqrt{2}/2, \sqrt{2}/2)$. Verify that $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are orthogonal matrices.

6. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find an orthonormal basis for the column space of $\boldsymbol{A}$.

(b) Write $\boldsymbol{A}$ as $\boldsymbol{QR}$.

(c) Compute the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(d) Find the least squares solution for $\boldsymbol{Ax} = \boldsymbol{c}$.

7. Consider the vector space $C[-2,2]$, the space of all real-valued continuous functions on $[-2,2]$, with inner product defined by

$$\langle f, g \rangle = \int_{-2}^{2} f(x)g(x)\,dx.$$

(a) Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

(b) Express x^2+2x as a linear combination of those orthonormal basis functions found in (a).

8. Consider the function

$$f(t) = \sin(2t)$$

on the interval from $-\pi$ to π.

(a) What is the closest function $a\cos t + b\sin t$ to $f(t)$ on the interval from $-\pi$ to π?

(b) What is the closest function $c + dt$ to $f(t)$ on the interval from $-\pi$ to π?