【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (10 points) Given a linear system

$$Ax=b, \text{ here } A=\begin{bmatrix}1&0&0&1\\1&1&1&4\\1&-2&4&1\\1&-1&1&0\\1&2&4&9\end{bmatrix} \text{ and } b=\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix},$$

Find all possible least square solutions.

2. (10 points) Find the inverse and the eigenvalues of the following matrix

$$A=\begin{bmatrix}4&-1&-1&-1\\-1&4&-1&-1\\-1&-1&4&-1\\-1&-1&-1&4\end{bmatrix}.$$

3. Let T_a be a linear transformation from $R^3 \to R^3$ defined as following $T_a(x)=a\times x$ where $x,\ a\in R^3$ and $a=(a_1,a_2,a_3)$ is an unit vector.

   (i) (8 points) Find the matrix-representation A of T_a.

   (ii) (7 points) Let

$$Q(t)=e^{At}=\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k \text{ for all } t\in R$$

   and $Q^T(t)$ be the transport matrix of $Q(t)$. Prove that $Q^T(1)=e^{-A}$.

4. Given

$$A=\begin{bmatrix}0&0.5&0.5\\0.5&0.5&0\\0.5&0&0.5\end{bmatrix}$$

   and a vector sequence $x_{n+1}=Ax_n,\ n=0,1,2,\cdots$.

   (i) (10 points) Prove that the sequence converges for any given initial vector $x_0\in R^3$.

   (ii) (5 points) Given

$$x_0=\begin{bmatrix}1\\0\\0\end{bmatrix},$$

   find $\lim_{n\to\infty} x_n$.

5. (10 points) Let A be the n×n matrix, $n \geq 3$,

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & \cdots & 0 & 5 \\ 5 & 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 5 & 3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 5 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 5 & 3 \end{bmatrix}.$$

Find det A.

6. (10 points) Suppose that A has the Jordan form

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the Jordan form of the matrix A^2.

7. (10 points) Let A be a real n×n matrix, $\langle Ax, x \rangle \geq 0$ for all $x \in R^n$. Show that $Au = 0$ if and only if $A^T u = 0$, where A^T is the transport of A.

8. Are the following statements true or false ? Give clear explanations of your answers or show counterexamples.

(i) (5 points) Every subset of R^n with more than n elements is a spanning set for R^n.

(ii) (5 points) If A is a real n×n matrix such that $A^2 = I$ then A is either I or -I, where I is the identity matrix.

(iii) (5 points) Let A be a 3×4 matrix, and B be a 4×3 matrix. Then det AB = 0.

(iv) (5 points) Let V be a vector space, and T: V → V be a linear transformation with T-invariant subspaces U, W. Then U+W and $U \cap W$ are also T-invariant subspaces.