# 國立交通大學 95 學年度碩士班考試入學試題

科目：線性代數(4061)　　考試日期：95 年 3 月 12 日 第 1 節

系所班別：應用數學系　　組別：應數所乙組在職生　　

＊＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！

**Notations.**

(1) The letter $\mathbb{R}$ denotes the set of real numbers. Hence, the notation $\mathbb{R}^n$ represents the usual Euclidean space of dimension $n$.

(2) The identity matrix of size $n$ is denoted by $I_n$.

(3) For a matrix $A$, we let $A^t$ denote the transpose of $A$, $\operatorname{tr} A$ the trace of $A$, and $|A|$ the determinant of $A$. For a nonsingular square matrix $B$, the notation $B^{-1}$ means the inverse of $B$.

(4) For a given vector space $\mathcal{V}$, the notation $\dim \mathcal{V}$ denotes the dimension of $\mathcal{V}$. If $S$ and $T$ are subspaces of $\mathcal{V}$, then $S+T$ denotes the subspace $\{u+v : u \in S, v \in T\}$.

(5) If T be a linear transformation, then KerT is the kernel of T, while Im T is the image of T.

(6) The notation $M_n(\mathbb{R})$ represents the set of all $n \times n$ matrices over $\mathbb{R}$.

**Problems.**

**1. (15 points.)** Let $\mathcal{U}$ be the solution space of

$$x_1 - x_2 + x_3 - x_4 = 0$$

in $\mathbb{R}^4$ and $\mathcal{V}$ be the solution space of

$$\begin{aligned} x_1 - 2x_2 \qquad + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - x_4 &= 0 \\ x_2 - x_3 - x_4 &= 0 \end{aligned}$$

in $\mathbb{R}^4$. Is there a linear transformation $\mathrm{T} : \mathbb{R}^4 \mapsto \mathbb{R}^4$ so that $\mathrm{T}u = u$ for all $u \in \mathcal{U}$ and $\mathrm{KerT} = \mathcal{V}$? If so, represent T in matrix with respect to a basis of your choice for $\mathbb{R}^4$. Justify your answer.

**2. (15 points.)** Let

$$B = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

Find *all* $3 \times 3$ real matrices $A$ such that $A^2 = B$. Justify your answer.

**3. (10 points.)** Let $A$ be a real $2 \times 2$ matrix with positive entries. Prove or disprove that there is an eigenvector $v$ of $A$ such that its components are all positive.

**4. (10 points.)** Prove that for $n \geq 2$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix} = \left( \sum_{j=1}^{n} x_j \right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

5. Let $\mathcal{V}$ be a vector space of finite dimension. Let $S$, $T$, and $U$ be vector subspaces of $\mathcal{V}$. Prove or disprove (by giving a counterexample) the following two formulas.
   (1) **(10 points.)** $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$.
   (2) **(10 points.)** $\dim(S+T+U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$.

6. (1) **(3 points.)** Prove that any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.
   (2) **(6 points.)** Let the linear transformation $\mathsf{T} : M_n(\mathbb{R}) \mapsto M_n(\mathbb{R})$ be defined by $\mathsf{T}(A) = A^t$. Determine the eigenvalues and eigenspaces of T.
   (3) **(6 points.)** Determine whether T is diagonalizable. If yes, diagonalize it; if not, prove it is not.

7. Let

$$A := \begin{pmatrix} 3 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

   (1) **(5 points.)** Find a matrix $P$ such that $P^{-1}AP$ is diagonal.
   (2) **(5 points.)** Find the maximum of $X^tAX$ among all $X \in \mathbb{R}^3$ subject to $X^tX = 1$. Give an example of $X$ that attains the maximum. Justify your answer.
   (3) **(5 points.)** Find the minimum of $\operatorname{tr}(Y^tAY)$ among all $3 \times 2$ matrices $Y$ subject to $Y^tY = I_2$. Give an example of $Y$ that attains the minimum. Justify your answer.