國立交通大學94學年度碩士班入學考試試題

v0201

科目名稱：線性代數(0191) 考試日期：94年4月16日 第1節

系所班別：應用數學系 組別：甲組,乙組 第 1 頁,共 2 頁

***作答前,請先核對試題、答案卷（試卷）與准考證上之所組別與考試科目是否相符!!**

Notations:

1. $F = R$ or C.
2. $F^{(n)}$ = the set of all column vectors $\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_n \end{bmatrix}$, where $a_i \in F$.
3. $M_n(F)$ = the set of all square matrices of dimension $n \times n$ with each elements in F.

1. (a) (5 points) State the definition of a basis $v_1, v_2, \cdots, v_n$ of $F^{(n)}$.
   (b) (5 points) Let T be a linear transformation on $F^{(n)}$ and $v_1, v_2, \cdots, v_n$ be a basis for $F^{(n)}$. Define the matrix of T in the basis $v_1, v_2, \cdots, v_n$.
   (c) (5 points) If you know the matrix A of a linear transformation T in the basis $v_1, v_2, \cdots, v_n$ of $F^{(n)}$. What is the matrix B of T in terms of A in the basis $v_n, v_{n-1}, \cdots, v_1$ of $F^{(n)}$?

2. (a) (5 points) Let $A \in M_n(F)$. State the definition of the minimal polynomial of A.
   (b) (5 points) Prove that the minimal polynomial of A is unique.
   (c) (5 points) Prove that every characteristic root (eigenvalue) of A is a root of the minimal polynomial of A.

3. (a) (5 points) State the definition of a subspace of $F^{(n)}$.
   (b) (5 points) For $A \in M_n(F)$, let V_a be the set $\{v \in F^{(n)} : (A - aI)^k v = 0$ for some positive integer k depending on $v\}$, where $a \in F$. Prove that V_a is a subspace of $F^{(n)}$.
   (c) (5 points) Let $v \in V_a$ and l be the first integer such that $(A - aI)^l v = 0$. Prove that $v, (A - aI)v, \cdots, (A - aI)^{l-1}v$ are linearly independent.
   (d) (5 points) If $a \neq b$ are in F, show that $V_a \bigcap V_b = \{0\}$.

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4. Determine whether the statement is true or false. If it is true, explain why. If it is false, give a counterexample.

   (a) (5 points) If A and B are $n \times n$ real matrices and $B \neq O$, then $\det(A + xB) = 0$ for some x in R.

   (b) (5 points) If A is an $n \times n$ real matrix, then the nullity of A equals the nullity of the transpose A' of A.

   (c) (5 points) For any $n \times n$ real matrix A, $A'A = AA'$, where A' is the transpose of A.

5. If V is a finite dimensional vector space, $T : V \to V$ is a linear transformation such that $T^3 - 3T^2 + 3T - I = O$, where $O : V \to V$, $O(v) = 0$ for all $v \in V$.

   (a) (5 points) Show that there is a $v \neq 0$ in V such that $T(v) = v$.

   (b) (5 points) Show that T is invertible.

   (c) (5 points) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Show that A is invertible, and express A^{-1} as a polynomial in A with real coefficients.

6. Let A be an $m \times n$ real matrix, and let A' be the transpose of A.

   (a) (5 points) Show that if $m = 2$ and $n = 4$, then the determinant of $A'A$ is 0.

   (b) (5 points) Write down certain conditions on A, m and n which will ensure that the determinant of $A'A$ is nonzero.

   (c) (5 points) Show that if $v_1, v_2, \cdots, v_n$ are linearly independent vectors of R^n, then the determinant of

$$\begin{bmatrix} (v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_n) \\ (v_2, v_1) & (v_2, v_2) & \cdots & (v_2, v_n) \\ \cdots & \cdots & \cdots & \cdots \\ (v_n, v_1) & (v_n, v_2) & \cdots & (v_n, v_n) \end{bmatrix}$$

   is positive, where $(\, , \,)$ is the standard inner product of R^n.

   (d) (5 points) Is there any relationship between the rank of $A'A$ and the rank of A ?