國立交通大學 93 學年度碩士班入學考試試題

科目名稱：線性代數(361)　考試日期：93 年 4 月 18 日 第 1 節

系所班別：應用數學系　　組別：乙組　　　第 一 頁，共 二 頁

＊作答前，請先核對試題、答案卷（試卷）與准考證上之所組別與考試科目是否相符!!

1. Let A, B be both $n \times n$ real matrices.
   (i) (2%) What is the definition of the exponential, e^A, of A?
   (ii) (6%) Does $e^{A+B} = e^A e^B$ always hold? If it does, then give a proof; otherwise, give a counterexample. If it is the latter case, find a sufficient condition such that the above statement is true.
   (iii) (10%) Compute e^A for A equal to the matrices

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

2. (i) (2%) What is the definition of $tr(A)$, the trace of a square matrix A?
   (ii) (3%) If A and B are both $n \times n$ real matrices, prove that $tr(AB) = tr(BA)$.
   (iii) (5%) Find the trace of $I + A + A^2 + \ldots + A^{28}$ for

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}.$$

3. Let A, B be both $n \times n$ real matrices and assume that A is invertible.
   (i) (5%) Show that AB has the same nullity and rank as B.
   (ii) (5%) What is the relation between the set of eigenvalues for AB and that for BA?

4. Let $V_1, V_2, \ldots, V_m$ be nonzero subspaces of R^n.
   (i) (2%) What is the meaning of saying that $V_1 + V_2 + \ldots + V_m$ is a direct sum?
   (ii) (5%) Show that $V_1 + V_2$ is a direct sum if and only if $V_1 \cap V_2 = \{\mathbf{0}\}$.
   (iii) (5%) If $V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = \{\mathbf{0}\}$, is the sum $V_1 + V_2 + V_3$ direct? Either give a proof or a counterexample.

5. Let V be the vector space of all polynomials $p(x)$ with degree at most two, and let $T : V \to V$ be the linear transformation $T(p(x)) = \frac{d}{dx}p(x)$.
   (i) (8%) Prove that $p_1(x) = -x + 1, p_2(x) = x + 1$ and $p_3(x) = x^2 + 1$ form a basis of V.
   (ii) (7%) Find the matrix of T in the basis $\{p_1, p_2, p_3\}$ of V in (i).

6. (i) (10%) If A and B are $n \times n$ complex matrices such that $x^* Ax = x^* Bx$ for all vectors x in C^n, then prove that $A = B$.

(ii) (5%) If A and B are $n \times n$ real matrices satisfying $x^t Ax = x^t Bx$ for all x in R^n, is it still true that $A = B$? Either give a proof or a counterexample.

7. (i) (5%) Prove that the eigenvalues of a Hermitian matrix ($A^* = A$) are all real.
(ii) (5%) What can you say about the eigenvalues of a unitary matrix ($A^*A = I$)? Prove your assertion.

8. Let the matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.$$

(i) (4%) Find the characteristic polynomial and minimal polynomial of A.
(ii) (4%) Use the minimal polynomial of A in (i) to express A^{100} as a linear combination of A and I.
(iii) (2%) Use (ii) to compute A^{100}.