***作答前,請先核對試題、答案卷（試卷）與准考證上之所組別與考試科目是否相符!!**

Notations: Let $R$ be the field of real numbers; $R^n$ the vector space of dimension $n$ over $R$; $P_n(R)$ the family of all polynomials with real coefficients and with degree at most $n$, and $M_n(R)$ the family of all $n\times n$ matrices over $R$.

1. (10 %) Are $U = \{(x,y,z)\in R^3 \,\|\, x+2y-3z| + |3x+2y-z| = 0\}$,

$$V = \left\{(x,y,z)\in R^3 \,\middle|\, \begin{bmatrix} x & y\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix} y\\ z\end{bmatrix} = 0\right\},$$

$$W = \left\{(x,y,z)\in R^3 \,\middle|\, \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix} x & y\\ y & z\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix} = 0\right\}$$

subspaces of $R^3$, justify your answer. Find their dimensions over $R$, give their geometric interpretations if they are.

2. (10%) Show that the polynomials $f(x)=1$, $g(x)=x-1$, and $h(x)=(x-1)^2$ form a base of $P_2(R)$ over $R$. Find the coordinate of the vector $\alpha(x)=2x^2-5x+6$ relative to this base.

3. (15%)

   a. (5%) Find the dimension of $U\cap W$ over $R$, where

   $$U=\{(x_1,x_2,x_3,x_4,x_5)\in R^5 \,|\, x_1-x_3-x_4=0, x_1+x_2-2x_3-x_4=0\}, \text{ and}$$

   $$W=\{(x_1,x_2,x_3,x_4,x_5)\in R^5 \,|\, x_2=x_3=x_4, x_1+x_5=0\}.$$

   b. (5%) If $T$ is a linear transformation from $R^2$ into $R^3$ such that $T(1,3)=(2,1,\alpha)$, and $T(2,1)=(3,\beta,6)$, determine $(\alpha,\beta)$ so that $T$ is not one to one.

   c. (5%) If the eigenvalues of $A\in M_2(R)$ are 0 and 1, and their corresponding eigenvectors are $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\-1\end{bmatrix}$ respectively, find the matrix $A$.

4. (15%) Let $T(f(x)) = f(2) - f(1)x + \frac{1}{2}f''(0)x^2$ be a function from $P_2(R)$ into itself. Find $T^{36}(x^2+x+1)$, $T^{89}(4x^2+19x+3)$ respectively; and then find a general expression for $T^n(f(x))$.

5. (12%)

   a. (2%) Given $A, B \in M_n(R)$, what does it mean to say that $A$ is similar to $B$?

   Write $A \sim B$ if $A$ is similar to $B$.

   b. (5%) Show that $\sim$ is an equivalence relation.

   c. (5%) Suppose that $A \sim B$. Is it true that $\det A = \det B$? Is it true that $A^k \sim B^k$ for all $k = 1,2,3,\cdots$? Answer true or false. If true, prove it. If false, give a counterexample with $n = 2$.

6. (14%) Suppose $A \in M_n(R)$ is orthogonal, namely $A^T A = I$. Here $A^T$ is the transpose of $A$.

   a. (3%) What are the possible values of $\det A$?

   b. (5%) Suppose that $Av = \alpha v$ for some nonzero vector $v$. What are all the possible values of the scalar $\alpha$? Explain with proof and examples.

   c. (6%) Decide if each of the matrices $A^{-1}, 2A, A^2$ is orthogonal, answer yes or no. If yes, prove it. If no, no proof is needed.

7. (10%) Let $A = (a_{ij}) \in M_n(R)$ with $\det A = 5$. Suppose that

   $b_{1j} = a_{2j},\ b_{2j} = a_{1j},\ b_{ij} = a_{ij}$ for all $i \geq 3, j \geq 1$.

   $c_{ij} = a_{ji}, d_{ij} = 3a_{ij}$ for all $i, j \geq 1$.

   $e_{i1} = e_{i2} = a_{i1},\ e_{ij} = a_{ij}$ for all $i \geq 1, j \geq 3$.

   $f_{1j} = 2a_{1j} + 3a_{2j},\ f_{ij} = a_{ij}$ for all $i \geq 2, j \geq 1$.

   Find the determinants of $B = (b_{ij}), C = (c_{ij}), D = (d_{ij}), E = (e_{ij}), F = (f_{ij})$. No proof is needed.

8. (14%) Let $V$ be a vector space over $R$ with basis $\{s, t, u, v, w\}$, and $T: V \to V$ a linear transformation. Suppose that *exactly* 3 vectors of $\{0, u, v, w, 2v+3w\}$ are in the image of $T$.

   a. (5%) What are the possible values of rank $T$? Explain.

   b. (9%) Suppose in addition that $T^2 = 0$. What is the nullity of $T$? Give a nonzero vector in the kernel of $T$. Explain.