1 (15 pts.) Let T be a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^n$.

(1) Show that there exists a unique $n \times n$ matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$.

(2) If A is invertible, show that T is one-to-one and onto.

(3) Let K be the kernel of T. Find the dimension of K in terms of the matrix A and verify your answer.

2 (20 pts.) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T([x, y, z]) = [x+y, x+z, y-z]$.

(1) Verify that T is a linear transformation.

(2) Find the standard matrix representation of T.

(3) Describe, in terms of the matrix A, the class of all matrix representations of T.

(4) Is every matrix representation of T diagonalizable? Justify your answer.

3 (15 pts.) Let A be an $n \times n$ real matrix.

(1) Prove that the eigenvalues of the matrix A are the same as the eigenvalues of A^T. Here A^T denotes the transpose of A.

(2) Show by a counterexample that an eigenvector of A need not be an eigenvector of A^T.

(3) Let v_1 and v_2 be eigenvectors of A with correspoining eigenvalues λ_1 and λ_2, respectively. Prove that, if $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are independent vectors.

4 (10 pts.) Let $y \neq 0$ be a fixed column vector in $\mathbb{R}^n$ and set $A = yy^T$. Then A is an $n \times n$ matrix.

(1) Show that $\lambda = 0$ is an eigenvalue of A of multiplicity $n-1$.

(2) What is the single nonzero eigenvalue ?

5 (15 pts.) Let W be a subspace of $\mathbb{R}^n$ and $dim(W) = k \leq n$.

(1) The set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is the orthogonal complement of W, and is denoted by $W^\perp$. Show that $W^\perp$ is a vector subspace. What is the dimension of $W^\perp$?

(2) Show that every vector b in $\mathbb{R}^n$ can be expressed uniquely in the form $b = b_W + b_{W^\perp}$ for b_W in W and $b_{W^\perp}$ in $W^\perp$. We call b_W the projection of b on W.

(3) For every vector $b \in \mathbb{R}^n$, show that the projection b_W of b on W is the unique vector in W that is cloest to b (that is, $w = b_W$ minimizes the Euclidean distance $\|b - w\|$ from b to W for all w in W.)

6 (15 pts.) Find an orthogonal diagonalization of the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

that is, find an orthogonal matrix C such that $C^{-1}AC$ is a diagonal matrix D.

7 (10 pts.) Assume that b is a column vector in $\mathbb{R}^n$ and A is a real, symmetric, positive definite matrix of order n. Define a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \frac{1}{2}x^T Ax - b^T x$. Show that the unique minimum of $f(x)$ is given by $x = A^{-1}b$. *(Hint: Consider $n = 1$ first.)*