國立交通大學八十五學年度碩士班入學考試試題

※作答前，請先核對試題、答案卷(試卷)與准考證上之所組別與考試科目是否相符!!

Linear Algebra

(20%) 1. Let V be the vector space consisting of all polynomials of degree at most two, and let $T : V \longrightarrow V$ be the linear transformation $T(p(x)) = p(x-1)$ for any polynomial p in V.

(a) Find the matrix representation A of T relative to the ordered basis $\{x^2, x, 1\}$. (5%)

(b) Find the matrix representation B of T relative to the ordered basis $\{x, x+1, x^2-1\}$. (5%)

(c) Prove that A and B are similar by exhibiting an invertible matrix X such that XA = BX. (10%)

(20%) 2. Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$.

(a) Find all the eigenvalues of A. (5%)

(b) For every eigenvalue, find its algebraic multiplicity. (5%)

(c) For every eigenvalue, find its geometric multiplicity. (5%)

(d) Use the results in (a), (b) and (c) to determine whether A is diagonalizable, that is, whether there is an invertible matrix X such that $X^{-1}AX$ is a diagonal matrix. If A is, find one such X. (5%)

(20%) 3. Let W be the subspace of R^4 spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Use the row-echelon form to help you with the computations in (a), (b) and (c) below.

(a) Find a basis for W. (5%)

(b) Find a basis for $W^{\perp}$, the orthogonal complement of W. (5%)

(c) Express $b = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in the bases of W and $W^{\perp}$. (5%)

(d) Find the projection of b on the subspace W. (5%)

(20%) 4. Let A be an $n \times n$ real matrix.

(a) Prove that A is orthogonal if and only if $Ax \cdot Ay = x \cdot y$ for all vectors x and y in R^n. (10%)

(b) Use (a) to prove that A is orthogonal if and only if $\|Ax\| = \|x\|$ for all vectors x in R^n. (10%)

(10%) 5. Let A and B be real square matrices.

(a) Prove that rank $AB \leq$ rank A by showing that the column space of AB is contained in the column space of A. (5%)

(b) Is rank $AB \leq$ rank B always true? If it is, give a proof; otherwise, give a counterexample. (5%)

(10%) 6. Find a Jordan canonical form of the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$