1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let $\beta = \{(1,0),(0,1)\}$ be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$.
   (a) (6%) Prove that $\gamma$ is a basis for $\mathbb{R}^3$.
   (b) (6%) Compute $[T]_\beta^\gamma$, the matrix representation of $T$ in the ordered bases $\beta$ and $\gamma$.

2. (12%) Let $V$ and $W$ be vector spaces over a field $F$. Let $T : V \to W$ be a function satisfying $T(x+y) = T(x) + T(y)$. If $F$ is the set of rational numbers, then prove or disprove that $T$ is a linear transformation from $V$ to $W$.

3. Let $V$ and $W$ be finite dimensional vector spaces and let $T : V \to W$ be linear and one-to-one.
   (a) (7%) Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for $V$. Prove that $\{T(v_i)\}_{i=1}^n$ is linearly independent.
   (b) (7%) Let the dimension of $V$ be denoted by $\dim(V)$. Prove that $\dim(V) = \dim(W)$.

4. (12%) Let $A$ be an $m \times n$ matrix with rank $k$. Suppose that $B$ is an $m \times m$ matrix such that $BA = O$, where $O$ is the zero matrix. Prove that $\text{rank}(B) \le m - k$.

5. Let $V$ be a vector space and let $T : V \to V$ be linear. If $B = \begin{bmatrix} O & A \\ A & O \end{bmatrix}$ is the matrix representation of $T$, where $O$ is the $n \times n$ zero matrix and $A$ is an $n \times n$ matrix whose entries are all equal to one, then
   (a) (5%) find the range of $T$;
   (b) (5%) find the nullity of $T$;
   (c) (5%) find all eigenvalues of $T$.

6. (15%) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, $I \in \mathbb{R}^{k\times k}$ be an identity matrix, and $U, V \in \mathbb{R}^{n\times k}$, where $k \le n$. Prove that if $I + V^\top A^{-1} U$ is invertible, then

$$(A + UV^\top)^{-1} = A^{-1} - A^{-1}U(I + V^\top A^{-1}U)^{-1}V^\top A^{-1}.$$

7. Let $A$ be an $n\times n$ anti-diagonal matrix (i.e., all the entries are zero except those on the diagonal going from the lower left corner to the upper right corner). If all anti-diagonal entries of $A$ are one, then
   (a) (5%) find the trace of $A$;
   (b) (5%) compute $A^2$;
   (c) (5%) find the minimal polynomial of $A$;
   (d) (5%) find the Jordan canonical form of $A$.