

國立交通大學 100 學年度碩士班考試入學試題

科目：線性代數(4042)

考試日期：100年2月18日 第2節

系所班別：應用數學系

組別：應數系乙組

第1頁，共2頁

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Let $A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$ be a 2×2 real matrix.

(a) (5 points) Compute A^{2011} .

(b) (5 points) Compute A^{-19} .

2. (10 points) Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 real matrix and suppose that A has an eigenvalue

$$\lambda = \max_{1 \leq i \leq 2} \left(\sum_{j=1}^2 \frac{|a_{ij}| + |a_{ji}|}{2} \right).$$

Prove that $A = A^T$, where A^T is the transpose of A .

3. Let $A = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \\ 3 & 0 & 6 \end{pmatrix}$ be a 4×3 real matrix.

(a) (10 points) If possible, find a 3×4 matrix B and a 4×3 matrix C such that $BA = I_3$ and $AC = I_4$, where I_k is the $k \times k$ identity matrix for $k = 3, 4$. If not, explain why.

(b) (10 points) Prove that $A^T A$ is diagonalizable with nonnegative eigenvalues $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq 0$, where A^T is the transpose of A and $\lambda_{j=1,2,3}$ are chosen to be nonnegative.

(c) (10 points) Let e_1, e_2, e_3 be the canonical basis of \mathbb{R}^3 . Prove or disprove that there is an orthonormal basis $\{v_j\}_{j=1}^4$ of \mathbb{R}^4 such that $Ae_j = \lambda_j v_j$ for $j = 1, 2, 3$. Here $\lambda_{j=1,2,3}$ is as above.

4. (10 points) Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be two inner product spaces and $T : V \rightarrow W$ be a linear transformation. We say T is an *isometry* if $\langle Tv_1, Tv_2 \rangle_W = \langle v_1, v_2 \rangle_V$.

Now let $V = \mathbb{R}^3$ equipped the standard inner product structure and $W = P_2[x]$, the space of all polynomials with real coefficients of degree less than 3 equipped with the standard vector space structure and the inner product structure defined by

$$\int_0^1 p_1(x)p_2(x) dx \quad \text{for all } p_1(x), p_2(x) \in P_2[x].$$

Let $T : \mathbb{R}^3 \rightarrow P_2[x]$ given by $T \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \alpha + \beta x + \gamma x^2$ be an isometry. Express α, β and γ in terms of a, b and c explicitly.

5. Let $M_n(\mathbb{R})$ be the collection of all $n \times n$ real matrices.

(a) (8 points) Let $A, B \in M_n(\mathbb{R})$. Prove that AB and BA have same eigenvalues.

(b) (10 points) Prove that if $A, B \in M_n(\mathbb{R})$ such that $AB = BA$, then A and B have a common eigenvector.

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- (c) (8 points) Let $A \in M_n(\mathbb{R})$ and let $p(t)$ is a polynomial function. Show that if A has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $\text{tr } p(A) = \sum_{i=1}^n p(\lambda_i)$, where $\text{tr } B$ is the trace of a matrix $B \in M_n(\mathbb{R})$.
- (d) (4 points) We say $A \in M_n(\mathbb{R})$ is *nilpotent* if, for some positive integer k , $A^k = 0$. Prove that if A is nilpotent, then $A^n = 0$.
- (e) (10 points) Let $A, B \in M_n(\mathbb{R})$ be nilpotent. Prove that if B has n distinct eigenvalues in \mathbb{R} and $AB = BA$, then $A = 0$.