1. (*20pts, 2pts for each*) For the following statements, please answer true or false. If false, please explain why.

   (a) Let X be a continuous random variable. For any x in $(-\infty, \infty)$, the probability that X equals x is zero.

   (b) Two random variables with zero correlation coefficient are independent.

   (c) Suppose that $X_1, \ldots, X_n$ are i.i.d. from Cauchy distribution. From the law of large number theory, we know that the variance of
   $$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$
   converges to zero when the sample size n goes to infinity.

   (d) A point *estimate* (i.e., an estimated value) of a parameter θ is a random variable with an associated probability distribution, called sampling distribution.

   (e) In the asymptotic method, the exact sampling distribution is often replaced by a Normal distribution while in the (parametric) bootstrap method, we often pretend that the estimated value of parameter is the true parameter.

   (f) Because a likelihood function represents the probability (or density) of observing data under different values of parameter θ, the summation (or integration) of a likelihood function over θ must be one.

   (g) For the case of an i.i.d. sample from a pdf $f(x|\theta)$ satisfying appropriate smooth assumptions, *no estimators* can have a smaller variance than the Cramer-Rao bound.

   (h) For the case of an i.i.d. sample from a pdf $f(x|\theta)$ satisfying appropriate smooth assumptions, the MLE is asymptotically efficient, i.e., the ratio between its variance and the Cramer-Rao bound equals one when the sample size tends to infinity.

   (i) Suppose that T is a sufficient statistic of $X_1, \ldots, X_n$, which follows a joint pdf $f(x_1, \ldots, x_n|\theta)$. Then, T contains *all* information in the data $X_1, \ldots, X_n$.

   (j) Suppose that $X_1, \ldots, X_n$ are i.i.d. from Normal distribution $N(\mu, \sigma^2)$, where μ and σ are unknown. Then, $n(\overline{X}_n - \mu)^2$ is an unbiased estimator of σ^2 because
   $$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \;\Rightarrow\; \frac{n(\overline{X}_n - \mu)^2}{\sigma^2} \sim \chi^2_1 \;\Rightarrow\; E[n(\overline{X}_n - \mu)^2] = \sigma^2.$$

2. (*18pts, 3pts for each*) For each of the random observations X below,

   - determine the type of its distribution (i.e., Normal, Exponential, Gamma, Beta, Uniform, Poisson, Hyper-geometric, Binomial, Bernoulli, Negative binomial, Geometric, etc.) which best models X, and
   - identify whether the parameters in the chosen distribution are known or unknown and give the values for known parameters.

(a) A player in the California lottery chooses 6 numbers from 1 to 53 and the lottery officials later choose 6 numbers at random. Let X equal the number of matches.

(b) Let X be the number of fatalities that resulted from being kicked by a horse recorded for 10 corps of Prussian cavalry in a year.

(c) A drunkard executes a "random walk" experiment in the following way: Each minute he takes a step north or south, with probability 1/2 each, and his successive step directions are independent. Suppose that the step lengths are about the same. After a *quite long* period of time, let X be the distance between his/her final and original locations. [**Hint**. Apply central limit theorem.]

(d) A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects that have been manufactured. Suppose that N is unknown. One of these objects is selected at random, and the serial number of that object is denoted as X.

(e) George tosses a coin three times. Let X_1 be the number of heads in the three tosses. He then gives the coin to Hilary. She tosses it until the first head occurs. Let X_2 be the number of times Hilary tosses. Set $X = (X_1, X_2)$.

(f) For two factors — starchy or sugary, and green base leaf or white base leaf — the following counts for the progeny of self-fertilized heterozygotes are observed:

Type	Count
Starchy gree	X_1
Starchy white	X_2
Sugary green	X_3
Sugary white	X_4

where $X_1+X_2+X_3+X_4 = n$ and n is fixed in advance. Set $X = (X_1, X_2, X_3, X_4)$.

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables from Uniform distribution $U(0, \theta)$ and set

$$Y_n = \max(X_1, \ldots, X_n) \quad \text{and} \quad Z_n = n(\theta - Y_n).$$

(a) (*6pts*) Prove that Y_n converges in probability to θ by showing that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|Y_n - \theta| < \epsilon) = 1.$$

(b) (*6pts*) Prove that Z_n converge in distribution to Z, where Z has the Exponential distribution $E(\lambda)$ with parameter $\lambda = 1/\theta$.
[**Hint**. **(i)** The pdf of $E(\lambda)$ is $\lambda e^{-\lambda x}$, for $x \geq 0$; the cdf of $E(\lambda)$ is $1 - e^{-\lambda x}$, for $x \geq 0$; the mgf of $E(\lambda)$ is $\frac{\lambda}{\lambda - t}$, for $t < \lambda$. **(ii)** $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$.]

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with the pdf

$$f(x|\theta) = (\theta + 1)x^\theta, \quad 0 \leq x \leq 1.$$

(a) (*6pts*) Find the MLE of θ.

(b) (*6pts*) Find the asymptotic variance and asymptotic sampling distribution of the MLE.

5. Let $Y_1, \ldots, Y_n$ be i.i.d. from the Uniform distribution $U(\theta, \theta+1)$.

   (a) (*4pts*) Find the method of moments estimator for θ (denoted as $\hat{\theta}_1$) and show that $\hat{\theta}_1$ is an unbiased estimator of θ.

   (b) (*2pts*) Find the standard error of $\hat{\theta}_1$.

   (c) (*6pts*) Let

   $$\hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1},$$

   where $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$. Show that $\hat{\theta}_2$ is an unbiased estimator of θ. [**Hint. (i)** Let $T_i = Y_i - \theta$, then T_i has a Uniform distribution $U(0,1)$. **(ii)** Let $T_{(n)} = \max\{T_1, \ldots, T_n\}$, then $T_{(n)} = Y_{(n)} - \theta$. Therefore $\mathrm{E}(T_{(n)}) = \mathrm{E}(Y_{(n)} - \theta)$ and $\mathrm{Var}(T_{(n)}) = \mathrm{Var}(Y_{(n)})$.]

   (d) (*4pts*) Find the standard error of $\hat{\theta}_2$.

   (e) (*2pts*) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$, denoted as $\mathrm{eff}_n(\hat{\theta}_1, \hat{\theta}_2)$, and find their asymptotic relative efficiency.

   (f) (*4pts*) When the sample size n is greater than 7, explain which of $\hat{\theta}_1$ and $\hat{\theta}_2$ is better in terms of mean square error. [**Hint.** $\mathrm{eff}_n(\hat{\theta}_1, \hat{\theta}_2) < 1$, when $n > 7$.]

6. Let $X_1, \ldots, X_r$ be an i.i.d. sample from binomial distribution $B(n,p)$, i.e., its pdf is:

   $$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad \text{where } x = 0, 1, \ldots, n, \text{ and } 0 \le p \le 1.$$

   We are interested in estimating the parameter $\theta = (1-p)^n$.

   (a) (*5pts*) Show that $T = \sum_{i=1}^{r} X_i$ is a sufficient and complete statistic for p.

   (b) (*3pts*) Show that

   $$U = \begin{cases} 1, & \text{if } X_1 = 0 \\ 0, & \text{otherwise} \end{cases}$$

   is an unbiased estimator.

   (c) (*5pts*) Show that

   $$P(U = 1|T = t) = \begin{cases} \dfrac{\binom{n(r-1)}{t}}{\binom{nr}{t}}, & \text{if } t \le n(r-1) \\ 0, & \text{if } t > n(r-1) \end{cases}$$

   (d) (*3pts*) Explain how to obtain a UMVUE of θ and find the UMVUE.

   [**Hint**: Let $Y_1, \ldots, Y_k$ be independent and $Y_i \sim B(n_i, p)$, $i = 1, \ldots, k$, then $Y_1 + \cdots + Y_k \sim B(n_1 + \cdots + n_k, p)$.]