(i)

57.

a). $\because \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = n-1 \Rightarrow E(S^2) = \sigma^2 \therefore S^2$ is unbiased

$E(\hat{\sigma}^2) = E\left(\frac{n-1}{n} S^2\right) = \frac{n-1}{n}\sigma^2 \Rightarrow \therefore \hat{\sigma}^2$ is ~~un~~biased

b). ①

$Var\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1) \Rightarrow Var(S^2) = \frac{2}{n-1}\sigma^4$.

$bias(S^2) = 0$.

$MSE(S^2) = \frac{2}{n-1}\sigma^4$.

②

$Var(\hat{\sigma}^2) = Var\left(\frac{n-1}{n} S^2\right) = \frac{2(n-1)}{n^2}\sigma^4$

$bias(\hat{\sigma}^2) = \frac{1}{n}\sigma^2$

$MSE(\hat{\sigma}^2) = \frac{1}{n^2}\sigma^4 + \frac{2(n-1)}{n^2}\sigma^4 = \frac{2n-1}{n^2}\sigma^4$

$\because \frac{2}{n-1} - \frac{2n-1}{n^2} = \frac{3n-1}{n^2(n-1)} > 0$

$\therefore \hat{\sigma}^2$ has smaller MSE.

c). Let $\tilde{\sigma}^2 = \rho \Sigma (X_i - \bar{X})^2 = \rho(n-1)S^2$

$Var(\tilde{\sigma}^2) = 2(n-1)\rho^2\sigma^4$

$bias(\tilde{\sigma}^2) = \rho(n-1)\sigma^2 - \sigma^2 = (\rho(n-1)-1)\sigma^2$

$MSE(\tilde{\sigma}^2) = \left([\rho(n-1)-1]^2 + 2(n-1)\rho^2\right)\sigma^4$

$$= [(n^2-1)\rho^2 - 2(n-1)\rho + 1]\,\sigma^4 .$$

$$\frac{\partial MSE(\hat{\sigma}^2)}{\partial \rho} = 2(n-1)(n+1)\rho - 2(n-1) \equiv 0 \Rightarrow \rho = \frac{1}{n+1}$$

$\tilde{\sigma}^2 = \frac{1}{n+1}\sum(X_i-\bar{X})^2$ has the minimal MSE.

(ii) 60.

e) by (a). $\hat{\tau}_{mle} = \bar{X}$.

$\because CRLB(\tau) = \frac{1}{nI(\tau)}$

$$I(\tau) = -E\left(\frac{\partial^2}{\partial\tau^2}\ln f(x|\tau)\right) = -E\left(\frac{1}{\tau^2} - \frac{2X}{\tau^3}\right).$$

$$= -\frac{1}{\tau^2} + \frac{2\tau}{\tau^3} = \frac{1}{\tau^2}$$

In regular condition, any unbiased of T.

$$Var(T) \geq \frac{1}{nI(\tau)} = \frac{\tau^2}{n} = CRLB(\tau).$$

and $Var(\hat{\tau}_{mle}) = Var(\bar{X}) \overset{\text{by (d)}}{=\!=} \frac{\tau^2}{n}$.

$\therefore$ Since $\hat{\tau}_{mle}$ achieves the lower bound. There is no other unbiased estimator with smaller variance.

(viii)

a) $L(\underset{\sim}{X}|\theta) = (\frac{1}{\sqrt{2\pi}})^n \exp(-\frac{1}{2}(\Sigma X_i^2 - 2n\theta\bar{X} + \theta^2)) \in$ 1-para.

exponential family.

$\because c(\theta) = n\theta$, $T(\underset{\sim}{X}) = \bar{X}$, $\therefore \bar{X}$ is complete sufficient statistic (c.s.s) of θ.

$E(\bar{X}^2) = (E(\bar{X}))^2 + Var(\bar{X}) = \theta^2 + \frac{1}{n}$ $\therefore E(\bar{X}^2 - \frac{1}{n}) = \theta^2$

$\because \bar{X}^2 - \frac{1}{n}$ 為 θ^2 unbiased estimator and the function of c.s.s. By Rao-Blackwell-Lehmann-Scheffe

$\therefore \bar{X}^2 - \frac{1}{n}$ is the UMVUE of θ^2

b) $Var(\bar{X}^2 - \frac{1}{n}) = Var(\bar{X}^2) = E(\bar{X}^4) - (E(\bar{X}^2))^2$

$M_{\bar{X}}(t) = e^{(\theta t + \frac{t^2}{2n})}$

$$\frac{\partial^4 M(t)}{\partial t^4} = e^{(\theta t + \frac{t^2}{2n})}\Big(\frac{1}{n^2} + (\theta + \frac{t}{n})\frac{4}{n^2} + (\theta + \frac{t}{n})^2\frac{4}{n} + (\theta + \frac{t}{n})\frac{2}{n} + (\theta + \frac{t}{n})^4\Big)\Big|_{t=0}$$

$$= \frac{1}{n^2} + \frac{4\theta}{n^2} + \frac{4}{n}\theta^2 + \frac{2}{n}\theta^3 + \theta^4$$

$\therefore Var(\bar{X}^2 - \frac{1}{n}) = \frac{2}{n}(\theta^3 + 2\theta^2 + 2\theta)$.

(iv)

(a)

$$f(y|p) = p^{y}(1-p)^{1-y}$$
$$= e^{y\ln p + (1-y)\ln(1-p)}$$
$$= e^{y\ln\left(\frac{p}{1-p}\right) + \ln(1-p)}$$

By LN. CH8. P53, Definition 6.16, $c(p)=\ln\left(\frac{p}{1-p}\right)$, $T(y)=y$, $d(p)=\ln(1-p)$, $S(y)=1$, is a exponential family.

For iid $Y_1, \cdots, Y_n$, $\sum_{i=1}^{n} Y_i$ is sufficient and complete for p.

(b)

$$E[T] = P[Y_1=1, Y_2=0] = P[Y_1=1]\cdot P[Y_2=0] = p\cdot(1-p) \quad \#$$

(c)

$$P[T=1 \mid W=w] = \frac{P[T=1, W=w]}{P[W=w]}$$
$$= \frac{P[Y_1=1, Y_2=0, \sum_{i=1}^{n} Y_i = w]}{P[\sum_{i=1}^{n} Y_i = w]}$$
$$= \frac{P[Y_1=1, Y_2=0, \sum_{i=3}^{n} Y_i = w-1]}{P[\sum_{i=1}^{n} Y_i = w]} = \frac{p(1-p)\binom{n-2}{w-1}p^{w-1}(1-p)^{n-w-1}}{\binom{n}{w}p^{w}(1-p)^{n-w}}$$
$$= \frac{w(n-w)}{n(n-1)}$$

補充: ($\sum_{i=1}^{n} Y_i \sim Bin(n, p)$, $\sum_{i=3}^{n} Y_i \sim Bin(n-2, p)$)

(d)

$$E[T|W=w] = 1\cdot P[T=1 \mid W=w] = \frac{w(n-w)}{n(n-1)} = \frac{w(n-w)}{n\cdot n}\cdot\frac{n}{n-1}$$
$$= \frac{n}{n-1}\left[\frac{w}{n}\left(1-\frac{w}{n}\right)\right]$$

$$\therefore E[T|W] = \frac{n}{n-1}\left[\frac{W}{n}\left(1-\frac{W}{n}\right)\right] = \frac{n}{n-1}\left[\bar{Y}(1-\bar{Y})\right], \quad W=\sum_{i=1}^{n} Y_i$$

$\because$ T is unbiased for $p(1-p)$ and $W=\sum_{i=1}^{n} Y_i$ is c.s.s for p $\therefore$ By LN. P.71 Thm 6.19 $E[T|W]$ is UMVUE for $p(1-p)$ #

U

(a) $f(y|\theta) = \frac{3y^2}{\theta^3} I_{(0,\theta)}(y)$, ∵ 變數和參數無法完全分離.

∴ The pdfs don't form an exponential family,

(b)

$$f(\underset{\sim}{y}|\theta) = \prod_{i=1}^{n} f(y_i|\theta)$$

$$= \frac{3^n \prod_{i=1}^{n} y_i^2}{\theta^{3n}} \underbrace{I_{(0,\theta)}(y_{(n)})}_{g(Y_{(n)},\theta)}$$

by factorization theorem $Y_{(n)}$ is ss for θ.

且 $F(y) = \int_0^y \frac{3t^2}{\theta^3}\,dt = \frac{y^3}{\theta^3}$

(c) $f_{Y_{(n)}}(y|\theta) = n\left[F(y)\right]^{n-1} f(y)$

$$= n\left(\frac{y^3}{\theta^3}\right)^{n-1}\frac{3y^2}{\theta^3} = \frac{3^n\, y^{3n-1}}{\theta^{3n}} \quad , 0 \le y \le \theta$$

(d) Let $T = Y_{(n)}$

$$f_T(t) = \frac{3n\,t^{3n-1}}{\theta^{3n}} \quad , 0 \le t \le \theta$$

Let

$$E(g(t)) = 0 \quad , \forall\ \theta \ge 0$$

$$\int_0^\theta g(t)\frac{3n\,t^{3n-1}}{\theta^{3n}}\,dt = 0$$

$$\Rightarrow \int_0^\theta g(t)\,t^{3n-1}\,dt = 0 \overset{\text{可看成}}{\Longrightarrow} \int_{\theta_1}^{\theta_2} g(t)\,t^{3n-1}\,dt = 0 \text{ for any } 0<\theta_1<\theta_2<\infty$$

$$\Rightarrow g(t)\,t^{3n-1} = 0.$$

$$\Rightarrow g(t)$$

∴ $T = X_{(n)}$ is complete.

(c)

$$E(T) = \int_0^\theta t \times \frac{3n t^{3n-1}}{\theta^{3n}} \, dt = \int_0^\theta \frac{3n t^{3n}}{\theta^{3n}} \, dt = \frac{3n}{\theta^{3n}} \int_0^\theta t^{3n} \, dt$$

$$= \frac{3n}{3n+1} \theta \Rightarrow \frac{3n+1}{3n} T = \frac{3n+1}{3n} Y_{(n)} \text{ is unbiase estimator of } \theta$$

且 $Y_{(n)}$ is css for θ.

By Rao-Blackwell and Lemann-scheffe theorem

$\frac{3n+1}{3n} Y_{(n)}$ is UMVUE of θ.