18

(d) $f(x|\alpha) = \dfrac{[x(1-x)^2]^{\alpha-1}(1-x)}{B(\alpha, 2\alpha)}$, where $B(\cdot)$ is a Beta function

$= \exp\{(\alpha-1)[\ln x + 2\ln(1-x)] + \ln(1-x) - \ln B(\alpha, 2\alpha)\}$

$= \exp\{c_1(\alpha)T_1(x) + S(x) + d(\alpha)\}$, 其中 $c_1(\alpha) = (\alpha-1)$
$T_1(x) = \ln x + 2\ln(1-x)$
$S(x) = \ln(1-x)$
$d(\alpha) = \ln B(\alpha, 2\alpha)$

$\Rightarrow$ pdf is a exponetial family.

$f(\underline{x}|\alpha) = \prod_{i=1}^{n} \exp[c_1(\alpha)T_1(x_i) + S(x_i) + d(\alpha)]\, I_{(0,1)}(x_i)$

$= \exp[c_1(\alpha)\sum_{i=1}^{n} T_1(x_i) + \sum_{i=1}^{n} S(x_i) + n\,d(\alpha)] \prod_{i=1}^{n} I_{(0,1)}(x_i)$

$\Rightarrow$ $\sum \ln X_i + 2\ln(1-X)$ is complete sufficient statistic for θ.

21.

(C)

(i) $f(\underline{x}|\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I_{(\theta,\infty)}(x_i) = e^{-\Sigma x_i + n\theta} I_{(\theta,\infty)}(x_{(1)})$

$$= \underbrace{\left(e^{n\theta} I_{[\theta,\infty]}(x_{(1)})\right)}_{g(x_{(1)},\,\theta)} \underbrace{\left(e^{-\Sigma x_i}\right)}_{h(\underline{x})}$$

By factorization theorem $X_{(1)}$ is a sufficient statistic of θ

(ii)

$f_{X_{(1)}}(x) = n\left[e^{-(x-\theta)}\right]^{n-1} e^{-(x-\theta)} = ne^{-n(x-\theta)}, \quad x \geq \theta$

Let u be a function s.t $E[u(X_{(1)})] = 0$ for all θ

Then $\int_{\theta}^{\infty} u(x)\, ne^{-n(x-\theta)}\, dx = 0, \quad \forall \theta$

$\Rightarrow e^{n\theta} \int_{\theta}^{\infty} u(x)\, ne^{-nx}\, dx = 0,$

here we know $\int_{\theta_1}^{\theta_2} u(x) ne^{-nx}\, dx = 0$ for any $0 < \theta_1 < \theta_2 < \infty$

$\Rightarrow u(x) = 0$ $\therefore$ $X_{(1)}$ is complete.

(iii)

$f(x|\theta) = e^{-(x-\theta)} I_{(\theta,\infty)}(x)$

發現隨机变数無法和參数完全分離.

$\therefore$ pdfs don't form an exponential family.

CM 0

49.

(a)

令 $Y_i = \begin{cases} 1 & \text{if } 0 < x_i < 1 \\ 0 & \text{if } -1 < x_i < 0 \end{cases}, \quad i = 1, 2, \cdots, n$

則 $Y_i \overset{iid}{\sim} Ber\,(p = P[0 < X < 1])$, $p = P[0 < X < 1] = \int_0^1 \frac{1+\alpha x}{2}\,dx = \frac{2+\alpha}{4}$

令 $\frac{d \ln L(y|\alpha)}{d\alpha} = \frac{\sum_{i=1}^{n} y_i}{2+\alpha} - \frac{n - \sum_{i=1}^{n} y_i}{2-\alpha} = 0 \Rightarrow$ 解得 $\hat{\alpha} = 4\bar{Y} - 2$, 且 $\frac{d^2 \ln L(y|\alpha)}{d\alpha^2} < 0$

$\therefore \hat{\alpha} = 4\bar{Y} - 2$ is MLE for α. ※

(b)

(i) $Var[\hat{\alpha}] = Var[4\bar{Y} - 2] = \frac{16}{n} \cdot p(1-p) = \frac{4-\alpha^2}{n}$

(ii) $E[X] = \int_{-1}^{1} \frac{x + \alpha x^2}{2}\,dx = \frac{\alpha}{3}$, 令 $\bar{X} \xrightarrow{est} E[X] \Rightarrow \hat{\alpha}_{MME} = 3\bar{X}$

$Var[\hat{\alpha}_{MME}] = Var[3\bar{X}] = \frac{9}{n} Var[X] = \frac{\cancel{9}}{n} \cdot \frac{3-\alpha^2}{\cancel{9}} = \frac{3-\alpha^2}{n}$

(iii)

$$Var[\hat{\alpha}_{MLE}] \approx \frac{1}{I_{X_1, X_2, \cdots X_n}(\alpha)}$$

$$I_{X_1, X_2, \cdots, X_n}(\alpha) = E\left[-\frac{d^2 \ln L(\alpha|\underline{x})}{d\alpha^2}\right] = E\left[\sum_{i=1}^{n} x_i^2 (1-\alpha x_i)^{-2}\right]$$

$$= n \cdot \int_{-1}^{1} \frac{x^2}{(1-\alpha x)} \cdot \frac{1}{2}\,dx$$

$$= \begin{cases} \dfrac{n\,(\ln(1+\alpha) - \ln(1-\alpha) - 2\alpha)}{2\alpha^3}, & \text{if } -1 \le \alpha \le 1,\ \alpha \ne 0 \\[2ex] \dfrac{n}{3}, & \text{if } \alpha = 0 \end{cases}$$

(By TB P.299)

(iv) $eff(\hat{\alpha}, \hat{\alpha}_{MME}) = \frac{Var[\hat{\alpha}_{MME}]}{Var[\hat{\alpha}]} = \frac{3-\alpha^2}{4-\alpha^2}$

$$eff(\hat{\alpha}, \hat{\alpha}_{MLE}) = \frac{Var[\hat{\alpha}_{MLE}]}{Var[\hat{\alpha}]} = \begin{cases} \dfrac{2\alpha^3}{(4-\alpha^2)(\ln(1+\alpha) - \ln(1-\alpha) - 2\alpha)}, & \text{if } -1 \le \alpha \le 1,\ \alpha \ne 0 \\[2ex] \dfrac{3}{4}, & \text{if } \alpha = 0 \end{cases}$$

※

(v)

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$eff(\hat{\alpha}, \hat{\alpha}_{MME})$	0.7500	0.7494	0.7475	0.7442	0.7396	0.7333	0.7253	0.7151	0.7024	0.6865
$eff(\hat{\alpha}, \hat{\alpha}_{MLE})$	0.7500	0.7474	0.7393	0.7254	0.7048	0.6960	0.6371	0.5841	0.5103	0.3994

∴ 由上表可知 $\hat{\alpha}$ 沒有比 $\hat{\alpha}_{MME}$ 和 $\hat{\alpha}_{MLE}$ 來得好，當 α 越接近1時，越明顯。

69.

$X_i \overset{iid}{\sim} Geo(p)$, $x_i = 1, 2, 3, \cdots$, $i = 1, 2, \cdots, n$

$$f(\underset{\sim}{x} \mid p) = (1-p)^{\sum_{i=1}^{n} x_i - n} \cdot p^n = (1-p)^{\sum_{i=1}^{n} x_i} \left(\frac{p}{1-p}\right)^n$$

By Thm A in Section 8.8.1 , $g\left(t = \sum_{i=1}^{n} x_i , p\right) = (1-p)^t \cdot \left(\frac{p}{1-p}\right)^n$, $h(\underset{\sim}{x}) = 1$

∴ $\sum_{i=1}^{n} X_i$ is sufficient statistic. ※

CH 8.

71.

$$f(\underset{\sim}{x}|\theta) = \frac{\theta^n}{\prod_{i=1}^{n}(1+x_i)^{\theta+1}}$$

$$= \exp\left(-(\theta+1)\sum_{i=1}^{n}\ln(1+x_i) + n\ln\theta\right) \in \text{1-parameter exponential family}$$

$c(\theta) = -(\theta+1)$, $T(\underset{\sim}{X}) = \sum_{i=1}^{n}\ln(1+x_i)$

$\therefore$ $\sum\ln(1+x_i)$ is complete and sufficient statistic (C.S.S) for θ.

72.

$$f(x/\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x} = \exp\left((\alpha-1)\ln x - \lambda x\right)\frac{\lambda^\alpha}{\Gamma(\alpha)}$$

$\in$ 2-parameter exponential family.

$c_1(\alpha,\lambda) = \alpha-1$, $c_2(\alpha,\lambda) = -\lambda$.

$t_1(x) = \ln x$, $t_2(x) = x$.

$T_1(\underset{\sim}{X}) = \sum\ln x_i = \ln\prod x_i$, $\because$ $\prod x_i$ is a 1-1 function of $\ln$.

$T_2(\underset{\sim}{X}) = \sum x_i$.

$\therefore$ $(\prod x_i, \sum x_i)$ are c.s.s. of (α,λ)