CH8

7.(c)

$\because \sqrt{nI(p)}(\hat{p}-p) \xrightarrow{d} N(0,1)$

$\hat{p} \xrightarrow{d} N(p, \frac{1}{nI(p)})$,where $I(p)$ is fisher information of X.

$$I(p) = E[\frac{\partial \ln f(X|p)}{\partial p}]^2 = E[-\frac{\partial^2 \ln f(X|p)}{\partial p^2}] = E[\frac{1}{p^2}+\frac{x-1}{(1-p)^2}](\because X \sim Geo(p), E(X) = \frac{1}{p})$$

$$= \frac{1}{p^2(1-p)}$$

$\therefore \hat{p} \xrightarrow{d} N(p, \frac{p^2(1-p)}{n}), AsyVar(\hat{p}) = \frac{p^2(1-p)}{n}$

16.(c)

$\because \sqrt{nI(\sigma)}(\hat{\sigma}-\sigma) \xrightarrow{d} N(0,1) \therefore \hat{\sigma} \xrightarrow{d} N(\sigma, \frac{1}{nI(\sigma)})$

$I(\sigma) = E[-\frac{\partial^2 \ln f(X|\sigma)}{\partial \sigma^2}] = E[-\frac{1}{\sigma^2} + \frac{2|x|}{\sigma^3}] = \frac{1}{\sigma^2}$

$(E|x| = \int_{-\infty}^{\infty} |x| \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}} dx = \int_0^{\infty} \frac{x}{\sigma} e^{\frac{-x}{\sigma}} dx = \sigma)$

$\hat{\sigma} \xrightarrow{d} N(\sigma, \frac{\sigma^2}{n}), AsyVar(\hat{\sigma}) = \frac{\sigma^2}{n}$

47.(a)

$\mu = E[X] = \int_{x_0}^{\infty} \theta x_0^{\theta} x^{-\theta} dx = \frac{\theta}{-\theta+1} x_0^{\theta} x^{-\theta+1}|_{x_0}^{\infty} = \frac{\theta x_0}{\theta-1}$

Let $\bar{X} \xrightarrow{est} \mu, \therefore \hat{\theta} = \frac{\bar{X}}{\bar{X}-x_0}$ is the MME of θ

47.(b)

$$L(\theta|\underset{\sim}{x}) = \theta^n x_0^{n\theta} \prod_{i=1}^{n} x_i^{-\theta-1}$$

$$\ln l(\theta) = \ln L(\theta|\underset{\sim}{x}) = n\ln\theta + n\theta\ln x_0 + \sum_{i=1}^{n} -(\theta+1)\ln x_i$$

Let $\frac{\partial \ln L(\theta|\underset{\sim}{x})}{\partial\theta} = \frac{n}{\theta} + n\ln x_0 + (-\sum_{i=1}^{n} \ln_i) = 0$

$=> \hat{\theta} = \frac{n}{\sum_{i=1}^{n}(\ln x_i - \ln x_0)}$

$\because \frac{\partial^2 \ln L(\theta|\underset{\sim}{x})}{\partial\theta^2} = \frac{-n}{\theta^2} < 0, \therefore \hat{\theta} = \frac{n}{\sum_{i=1}^{n}(\ln x_i - \ln x_0)}$ is the MLE of θ

47.(c)

By CH8 LNp.40 note1

$$I_{X_1,\ldots,X_n}(\theta) = E[-\frac{\partial^2 \ln l(\theta|\underset{\sim}{x})}{\partial\theta^2}] = E[\frac{n}{\theta^2}] = \frac{n}{\theta^2}$$

$\therefore$ the asymptotic variance of $\hat{\theta}_{mle}$ is $\frac{1}{I_{X_1,\ldots,X_n}(\theta)} = \frac{\theta^2}{n}$

and the asymptotic distribution of $\hat{\theta}_{mle}$ is $\hat{\theta}_{mle} \xrightarrow{d} N(\theta, \frac{\theta^2}{n})$

50.(c)

$\because \sqrt{nI(\theta)}(\hat{\theta}-\theta) \xrightarrow{d} N(0,1), \hat{\theta} \xrightarrow{d} N(\theta, \frac{1}{nI(\theta)})$, where $I(\theta)$ is fisher information of X.

$$\ln f(x|\theta) = \ln x - 2\ln\theta - \frac{x^2}{2\theta^2}$$

$$\frac{\partial \ln f(x|\theta)}{\partial\theta} = \frac{-2}{\theta} + x^2\theta^{-3} = 0 => \hat{\theta} = \sqrt{\frac{x^2}{2}}$$

$$\frac{\partial^2 \ln f(x|\theta)}{\partial\theta^2} = \frac{2}{\theta^2} + -3x^2\theta^{-4} = \frac{1}{\theta^2}(2 - \frac{3x^2}{\theta^2})$$

$$I(\theta) = E[-\frac{\partial^2 \ln f(X|\theta)}{\partial\theta^2}] = E[-\frac{2}{\theta^2} + \frac{3X^2}{\theta^4}] = \frac{4}{\theta^2}$$

$(E[X^2] = \int_0^\infty \frac{x^3}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx$,Let $u = \frac{x^2}{2\theta^2}, du = \frac{x}{\theta^2}dx = \int_0^\infty 2ue^{-u}\theta^2 du = 2\theta^2)$

$\therefore$ the Fisher information of $X_1, X_2, ..., X_n$ is $nI(\theta) = \frac{4n}{\theta^2}$

and the asymptotic distributon of $\hat{\theta}_{mle} \sim N(\theta, \frac{\theta^2}{4n})$

58.(a)

Let X_1, X_2, X_3 be the counts of the following genotypes AA,Aa,and aa in n people.

$(X_1, X_2, X_3) \sim Multinomial(n, (1-\theta^2), 2\theta(1-\theta), \theta^2)$, where $\sum_{i=1}^{3} x_i = n$

$$f(x_1, x_2, x_3|\theta) = \frac{n!}{x_1!x_2!x_3!}((1-\theta)^2)^{x_1}(2\theta(1-\theta))^{x_2}(\theta^2)^{x_3}$$

$l(\theta) = \ln f(x_1, x_2, x_3|\theta) = -\ln(x_1!x_2!x_3!) + (2x_1+x_2)\ln(1-\theta) + (x_2+2x_3)\ln\theta + x_2\ln 2$

$$\frac{\partial l(\theta)}{\partial\theta} = -\frac{2x_1+x_2}{1-\theta} + \frac{x_2+2x_3}{\theta} \equiv 0 => \hat{\theta} = \frac{x_2+2x_3}{2n}$$

$$\frac{\partial^2 l(\theta)}{\partial\theta^2} = -\frac{2x_1+x_2}{(1-\theta)^2} - \frac{x_2+2x_3}{\theta^2} < 0$$

$\therefore \hat{\theta}_{mle} = \frac{x_2+2x_3}{2n} = \frac{68+2\cdot 112}{2\cdot 190} \approx 0.768.$

58.(b)

$$I(\theta) = E[-\frac{\partial^2 l(\theta)}{\partial\theta^2}] = E[\frac{2x_1+x_2}{(1-\theta)^2} - \frac{x_2+2x_3}{(\theta)^2}]$$

$$\left(\because \begin{array}{c} X_1 \sim Bin(n, (1-\theta)^2) \\ X_2 \sim Bin(n, 2\theta(1-\theta)) \\ X_3 \sim Bin(n, \theta^2) \end{array} \right)$$

$\therefore I(\theta) = \frac{2n}{\theta(1-\theta)}$, the asymptotic distribution of $\hat{\theta}_{mle} \xrightarrow{d} N(\theta, \frac{\theta(1-\theta)}{2n})$.
The asymptotic variance of $\hat{\theta}_{mle}$ is $\frac{\theta(1-\theta)}{2n}$.
The estimated asymptotic variance is $Var(\hat{\theta}_{mle}) = \frac{0.768(1-0.768)}{2\cdot 190} \approx 0.00047$.