Ch8

7.

(a) $X_1, X_2, \cdots, X_n \overset{iid}{\sim} Geo(p)$

$\because E(X_1) = \frac{1}{p} = \mu_1 \Rightarrow p = \frac{1}{\mu_1}$

$\hat{\mu}_1 = \frac{1}{n}\sum X_i \xrightarrow{estimate} \mu_1$

$\therefore \hat{p} = \frac{1}{\hat{\mu}_1} = \frac{n}{\sum X_i}$

(b)

$L(p|\underline{x}) = p^n (1-p)^{\sum X_i - n}$

$l(p|\underline{x}) = \ln(L(p|\underline{x})) = n\ln p + (\sum X_i - n)\ln(1-p)$

let $\frac{\partial}{\partial p} l(p|\underline{x}) = 0$

$\Rightarrow \frac{n}{p} - \frac{\sum X_i - n}{1-p} = 0 \Rightarrow \hat{p} = \frac{n}{\sum X_i}$ is mle of p

且 $\frac{\partial^2}{\partial p^2} l(p|\underline{x}) = -\frac{n}{p^2} - \frac{\sum X_i - n}{(1-p)^2}$ 恆小於 0.

9. $\hat{\lambda} = \overline{X}$ $\quad \overline{X} = \frac{1}{n}\sum X_i = 24.9$

$\overline{X}$ 為 estimator，它本身為一个 random variable，存在一个分配

$\frac{1}{n}\sum X_i = 24.9$ 則是 estimate，需要抽取樣本才能得到，会是一个 fixed value

基本上 estimate 可視為 estimator 的一个 realization

8.16

$$\mu_1 = E(X) = \int_{-\infty}^{\infty} \frac{1}{2\sigma} x e^{-\frac{|x|}{\sigma}} dx$$

$$= \int_0^{\infty} \frac{x}{2\sigma} e^{-\frac{x}{\sigma}} + \int_{-\infty}^{0} \frac{x}{2\sigma} e^{-\frac{x}{\sigma}} dx = 0.$$

再看2階动量

$$\mu_2 = E(X^2) = \int_{-\infty}^{\infty} \frac{x^2}{2\sigma} e^{-\frac{|x|}{\sigma}} dx$$

$$= \int_0^{\infty} \frac{x^2}{2\sigma} e^{-\frac{x}{\sigma}} dx + \int_{-\infty}^{0} \frac{x^2}{2\sigma} e^{-\frac{x}{\sigma}} dx$$

$$= \int_0^{\infty} \frac{x^2}{\sigma} e^{-\frac{x}{\sigma}} dx = \frac{1}{\sigma} \cdot \Gamma(3)\, \sigma^3 = 2\sigma^2 \Rightarrow \sigma = \sqrt{\frac{\mu_2}{2}}$$

$$\hat{\mu}_2 = \frac{1}{n}\sum X_i^2 \xrightarrow{\text{estimate}} \mu_2$$

$$\therefore \hat{\sigma} = \sqrt{\frac{\hat{\mu}_2}{2}} = \sqrt{\frac{\sum X_i^2}{2n}}.$$

(b) $L(\sigma|\underline{x}) = \frac{1}{2^n}\sigma^{-n} \exp\left(-\frac{\sum|X_i|}{\sigma}\right), \quad 0<\sigma<\infty$

$$l(\sigma|\underline{x}) = \ln(L(\sigma|\underline{x})) = -n\ln 2 - n\ln\sigma - \frac{\sum|X_i|}{\sigma}$$

Let $\frac{\partial}{\partial\sigma} l(\sigma|\underline{x}) = 0.$

$$\Rightarrow -\frac{n}{\sigma} + \frac{\sum|X_i|}{\sigma^2} = 0.$$

$$\Rightarrow \hat{\sigma} = \frac{\sum|X_i|}{n} \text{ is mle of } \sigma$$

且 $\frac{\partial^2}{\partial\sigma^2} l(\sigma|x) = \frac{n}{\sigma^2} - 2\frac{\sum|X_i|}{\sigma^3} \underset{\text{when } \sigma = \frac{\sum|X_i|}{n}}{=} -\frac{n^3}{(\sum|X_i|)^2} < 0.$

CH 8

19.

(a) $X_i \overset{iid}{\sim} N(u, \sigma^2)$, $i=1,2,\cdots,n$, u is known.

$$L(\sigma^2|\underset{\sim}{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-u)^2}{2\sigma^2}}$$

$$\ln L(\sigma^2|\underset{\sim}{x}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-u)^2}{2\sigma^2}$$

令 $\frac{d \ln L(\sigma^2|\underset{\sim}{x})}{d\sigma} = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-u)^2}{\sigma^3} = 0$, $\left.\frac{d^2 \ln L(\sigma^2|\underset{\sim}{x})}{d\sigma^2}\right|_{\hat{\sigma}} < 0$

解得 $\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(x_i-u)^2}{n}}$ 為 σ 之 MLE ※

(b) σ is known

令 $\frac{d \ln L(u|\underset{\sim}{x})}{du} = \frac{\sum_{i=1}^{n}(x_i-u)}{\sigma^2} = 0$, $\left.\frac{d^2 \ln L(u|\underset{\sim}{x})}{du^2}\right|_{\hat{u}} < 0$

解得 $\hat{u} = \frac{\sum_{i=1}^{n}x_i}{n} = \overline{X}$ 為 u 之 MLE ※

21.

(a) 令 $Y = X-\theta$, $Y \sim EXP(1)$, $y>0$

$\because E[Y] = E[X-\theta] = 1 \Rightarrow u_1 = E[X] = 1+\theta \Rightarrow \theta = u_1 - 1$

Let $\overline{X} \xrightarrow{estimate} u_1$, $\therefore \hat{\theta} = \overline{X} - 1$ 為 θ 之 MME ※

(b) $L(\theta|\underset{\sim}{x}) = e^{-\sum_{i=1}^{n}(x_i-\theta)} \cdot \prod_{i=1}^{n} \mathbb{1}_{(\theta,\infty)}(x_i)$

$= e^{-\sum_{i=1}^{n}(x_i-\theta)} \cdot \mathbb{1}_{(-\infty,\min(x_i))}(\theta)$ ← is an increasing function of θ over $(-\infty, \min(x_i))$

$\therefore$ 當 $\theta = \min(x_i)$ 時, $L(\theta|\underset{\sim}{x})$ 有最大值, $\therefore \hat{\theta} = \min(x_i)$ 為 θ 之 MLE ※

CH 8

26. 令母體數為 N

已標記動物數為 $T=100$,（assumption 不管怎麼選取，每個動物被選取的機率相同 機率之下，Hypergeometric 假設才成立）

樣本大小 $n=50$

樣本標記數 $x=20$

則 $X \sim HG(N, T, n)$

$$L(N|x)=f_X(x)=\frac{\binom{T}{x}\binom{N-T}{n-x}}{\binom{N}{n}}, \quad x=0,1,2,\cdots,n$$

令 $\hat{N}$ 為 N 之 MLE，則 $L(\hat{N}|x)$ 有最大值

則滿足 $\dfrac{L(\hat{N}|x)}{L(\hat{N}-1|x)} \geq 1$ 且 $\dfrac{L(\hat{N}|x)}{L(\hat{N}+1|x)} \geq 1$

解得：$\begin{cases} \hat{N} \leq \dfrac{Tn}{x} \\ \hat{N} \geq \dfrac{Tn}{x}-1 \end{cases} \Rightarrow \begin{cases} \hat{N} \leq \dfrac{100\times 50}{20} = 250 \\ \hat{N} \geq \dfrac{100\times 50}{20}-1 = 249 \end{cases}$

$\therefore$ MLE $\hat{N} = 249$ or 250 ※

(i) $$\frac{\cancel{\binom{T}{x}}\binom{\hat{N}-T}{n-x}}{\binom{\hat{N}}{n}} \times \frac{\binom{\hat{N}-1}{n}}{\cancel{\binom{T}{x}}\binom{\hat{N}-1-T}{n-x}} \geq 1$$

(ii) $$\frac{\cancel{\binom{T}{x}}\binom{\hat{N}-T}{n-x}}{\binom{\hat{N}}{n}} \times \frac{\binom{\hat{N}+1}{n}}{\cancel{\binom{T}{x}}\binom{\hat{N}+1-T}{n-x}} \geq 1$$

$$\Rightarrow \begin{cases} \dfrac{(\hat{N}-T)(\hat{N}-n)}{(\hat{N}-T-n+x)\hat{N}} \geq 1 \\ \dfrac{(\hat{N}+1)(\hat{N}+1-T-n+x)}{(\hat{N}+1-n)(\hat{N}+1-T)} \geq 1 \end{cases}$$

解得

CH 8.

5D.

a) $E(X) = \int_0^\infty \frac{x^2}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx = \mu_1$

(令 $u = \frac{x^2}{2\theta^2}$, $\left|\frac{du}{dx}\right| = \frac{\theta^2}{x}$ $|J| = \left|\frac{dx}{du}\right| = \frac{\theta}{\sqrt{2u}}$)

$$= \int_0^\infty 2u e^{-u} \times \frac{\theta}{\sqrt{2u}} du.$$

$$= \sqrt{2}\theta \int_0^\infty \sqrt{u} e^{-u} du.$$

$$= \sqrt{2}\theta\, \Gamma\left(\frac{3}{2}\right)$$

($Y \sim N(0,1)$. $f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, $-\infty < y < \infty$

令 $X = \frac{y^2}{2}$, $y = \sqrt{2x}$, $\left|\frac{dy}{dx}\right| = \frac{1}{\sqrt{2x}} = |J|$

$\therefore f(x) = \frac{1}{\sqrt{2\pi}} e^{-x} \frac{1}{\sqrt{2x}} \times 2$, $0 < x < \infty$.

$\because \int_0^\infty f(x) dx = 1 \Rightarrow \int_0^\infty x^{-\frac{1}{2}} e^{-x} = \sqrt{\pi} \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$)

$$= \sqrt{2}\theta \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{2}}\theta \Rightarrow \theta = \sqrt{\frac{2}{\pi}}\mu_1, \quad \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{\text{估計}} \mu_1$$

$$\tilde{\theta}_{MME} = \sqrt{\frac{2}{\pi}}\, \bar{X}.$$

b).

$$L(\underset{\sim}{x}|\theta) = \left(\frac{1}{\theta^2}\right)^n \prod_{i=1}^n x_i\, e^{-\frac{\sum_{i=1}^n x_i^2}{2\theta^2}}, \quad x_1, \dots, x_n \geq 0.$$

$$\ell(\underset{\sim}{x}|\theta) = \log L(\underset{\sim}{x}|\theta) = -2n\log\theta + \sum \ln x_i - \frac{\sum x_i^2}{2\theta^2}$$

$$\Rightarrow \frac{\partial \ell}{\partial\theta} = \frac{-2n}{\theta} - \frac{\sum x_i^2}{2}\left(\frac{-2}{\theta^3}\right) = 0 \Rightarrow \hat{\theta} = \sqrt{\frac{\sum x_i^2}{2n}}$$

$$\because \left.\frac{\partial^2 \ell}{\partial\theta^2}\right|_{\hat\theta} = \frac{1}{\theta^2}\left(2n - \frac{3\sum x_i^2}{\theta^2}\right)\Big|_{\hat\theta} = \frac{1}{\hat\theta^2}(-3n) < 0. \quad \therefore \hat{\theta}_{MLE} = \sqrt{\frac{\sum x_i^2}{2n}}$$

51.

$$L(\underline{x}|\theta) = \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^{n}|x_i-\theta|}, \quad -\infty < x_i < \infty$$

$$\Rightarrow \max_{\theta} L(\theta|\underline{x}) \Leftrightarrow \min_{\theta} \sum_{i=1}^{n} |x_i-\theta| \Leftrightarrow \min_{\theta} \sum_{i=1}^{n} |x_{(i)}-\theta|.$$

consider:

① $x_{(m)} \le \theta \le x_{(m+2)}$, $\sum_{i=1}^{n} |x_{(i)}-\theta| = |x_{(m+1)}-\theta| + x_{(m+2)} - x_{(m-1)} + \cdots + x_{(2m+1)} - x_{(1)}$

② $x_{(m-1)} \le \theta \le x_{(m+1)}$, $\sum_{i=1}^{n} |x_{(i)}-\theta| = |x_{(m)}-\theta| + x_{(m+1)}-\theta + x_{(m+2)}-\theta + x_{(m+3)} - x_{(m-1)} + \cdots + x_{(2m+1)} - x_{(1)}$

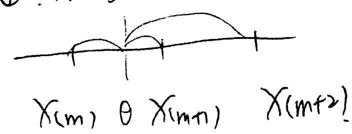
③ $x_{(m+1)} \le \theta \le x_{(m+3)}$, $\sum_{i=1}^{n} |x_{(i)}-\theta| = |x_{(m+2)}-\theta| + \theta - x_{(m)} + \theta - x_{(m+1)} + x_{(m+3)} - x_{(m-1)} + \cdots + x_{(2m+1)} - x_{(1)}$

同理

⋮

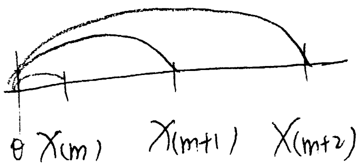
$x_{(j-1)} \le \theta \le x_{(j)}$, $\sum_{i=j}^{n} x_{(i)} - \sum_{i=1}^{j-1} x_{(i)} + (2j-2-n)\theta$.

由①與②比



①至多為 $\max(|x_{(m+2)} - x_{(m+1)}|, |x_{(m+1)} - x_{(m)}|) + x_{(m+2)} - x_{(m)} + \cdots + x_{(2m+1)} - x_{(1)}$

②至少為 $x_{(m+1)} - x_{(m)} + x_{(m+2)} - x_{(m)} + \cdots + x_{(2m+1)} - x_{(1)}$



∴同理可得①與③比

③至少為 $x_{(m+2)} - x_{(m+1)} + x_{(m+2)} - x_{(m)} + \cdots + x_{(2m+1)} - x_{(1)}$

由此可知在①的情況下,

$$\hat{\theta}_{MLE} = x_{(m+1)}, \quad L(\underline{x}|\theta) = \left(\frac{1}{2}\right)^n e^{-(x_{(m+2)} - x_{(m)} + \cdots + x_{(2m+1)} - x_{(1)})}$$ #.

60. $X_1, \cdots, X_n \sim \Gamma(1, \lambda = \frac{1}{\tau})$, $x_i > 0$, $i = 1, \cdots, n$

a).

$L(\underline{x}|\tau) = (\frac{1}{\tau})^n e^{-\frac{1}{\tau}\Sigma x_i}$

$\ell(\underline{x}|\tau) = -n\log\tau - \frac{1}{\tau}\Sigma x_i$

$\frac{\partial \ell(\underline{x}|\tau)}{\partial\tau} = \frac{-n}{\tau} + \frac{\Sigma x_i}{\tau^2} = 0 \Rightarrow \hat{\tau} = \frac{\Sigma x_i}{n} = \bar{x}$,

$\because \left.\frac{\partial^2 \ell(\underline{x}|\tau)}{\partial\tau^2}\right|_{\tau=\hat{\tau}} = \frac{1}{\hat{\tau}^2}(-n) < 0$. $\therefore \hat{\tau}_{MLE} = \bar{X}$.

b).

$X_i \sim \Gamma(1, \frac{1}{\tau})$

Gamma 加成性 $\therefore \Sigma X_i \sim \Gamma(n, \frac{1}{\tau})$

$\frac{1}{n}$為 scale parameter. $\frac{\Sigma X_i}{n} = \bar{X} \sim \Gamma(n, \frac{n}{\tau})$.

c).

By CLT, $\because E(\bar{X}) = \tau$, $Var(\bar{X}) = \frac{\tau^2}{n}$

$\therefore \frac{\sqrt{n}(\bar{X}-\tau)}{\tau} \xrightarrow{d} N(0,1) \Rightarrow \bar{X} \xrightarrow{d} N(\tau, \frac{\tau^2}{n})$.

d).

$E(\hat{\tau}_{MLE}) = E(\bar{X}) = \tau$.

$\because \bar{X} \sim \Gamma(n, \frac{n}{\tau})$

$\therefore Var(\bar{X}) = \frac{n}{(\frac{n}{\tau})^2} = \frac{\tau^2}{n}$ #