Ch5

①

$$E(\overline{X}_n) = \mu, \quad Var(\overline{X}_n) = \frac{\sum Var(X_i)}{n^2} = \frac{\sum \sigma_i^2}{n^2}$$

$$P(|\overline{X}_n - \mu| \geq \varepsilon) \leq E\left(\frac{|\overline{X}-\mu|^2}{\varepsilon}\right) = \frac{Var(\overline{X}_n)}{\varepsilon}$$

$\because$ $n^{-2}\sum \sigma_i^2 \longrightarrow 0$, so $Var(\overline{X}_n) \to 0$ as $n \to \infty$

$\therefore$ $P(|\overline{X}_n - \mu| \geq \varepsilon) \longrightarrow 0$.

i.e. $\overline{X}_n \xrightarrow{P} \mu$

③

$N \sim Poi(\lambda)$

$\because$ $\frac{N-\lambda}{\sqrt{\lambda}} \xrightarrow[\lambda\to\infty]{d} N(0,1)$ (LN p.95)

$$\therefore P(N > 10{,}200) = P\left(\frac{N-10{,}000}{\sqrt{10{,}000}} > \frac{10{,}200-10{,}000}{\sqrt{10{,}000}}\right)$$

$$\approx P(Z > 2) = 1 - 0.9772 = 0.0228,$$

where $Z \sim N(0,1)$

(5)

$X \sim Bin(n, p)$

$Y \sim poi(\lambda)$

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum_{x=0}^{\infty} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$

$= (1-p+e^t p)^n = \left(1 + \frac{np(e^t-1)}{n}\right)^n$

$p \to 0$, $n \to \infty$, $np \to \lambda$ $\left(1 + \frac{np(e^t-1)}{n}\right)^n \to e^{\lambda(e^t-1)} = M_Y(t)$

By the continuity Thm (LNp.94),

$$X \xrightarrow[\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}]{d} Y$$

ch 5

(10)

(i) 令 X 為成功投一次骰子為 6 的個數

$X_i \overset{iid}{\sim} Ber(\frac{1}{6})$, $i=1,2,\cdots,100$, $x=0,1$

$\mu = E[X_i] = \frac{1}{6}$, $\sigma^2 = Var[X_i] = \frac{5}{36}$

$\therefore \sum_{i=1}^{n} X_i \xrightarrow{d} N(n\mu, n\sigma^2)$ by CLT、

$\Rightarrow \sum_{i=1}^{100} X_i \xrightarrow{d} N(\frac{100}{6}, \frac{125}{9})$ by CLT、

所求 $P_r[15 < \sum_{i=1}^{100} X_i < 20] \approx P_r\left[\frac{15-\frac{100}{6}}{\sqrt{\frac{125}{9}}} < Z < \frac{20-\frac{100}{6}}{\sqrt{\frac{125}{9}}}\right]$ where $Z \sim N(0,1)$

$= \Phi(\frac{2}{\sqrt{5}}) - \Phi(-\frac{1}{\sqrt{5}}) = 0.487$ ※

(ii) 令 Y 為投一次骰子出現的點數

$Y_i \overset{iid}{\sim} f_Y(y) = \frac{1}{6}$, $y = 1, 2, 3, 4, 5, 6$, $i=1,2,\cdots,100$

$\mu = E[Y_i] = \frac{7}{2}$, $\sigma^2 = Var[Y_i] = \frac{35}{12}$

$\therefore \sum_{i=1}^{n} Y_i \xrightarrow{d} N[n\mu, n\sigma^2]$ by CLT

$\Rightarrow \sum_{i=1}^{100} Y_i \xrightarrow{d} N(350, \frac{25\times 35}{3})$ by CLT

所求 $P_r[\sum_{i=1}^{100} Y_i < 300] \approx P_r[Z < \frac{300-350}{5\sqrt{\frac{35}{3}}}] = \Phi[-\frac{10}{\sqrt{\frac{35}{3}}}]$

where $Z \sim N(0,1)$

$= \Phi[-\sqrt{\frac{60}{7}}]$

$= 0.0017$ ※

(16)

$X_i \overset{iid}{\sim} f(x) = 2x, \quad 0 \le x \le 1, \quad i = 1, 2, \cdots, 20$

$E[X] = \int_0^1 2x^2\,dx = \frac{2}{3}, \quad E(X^2) = \int_0^1 2x^3\,dx = \frac{1}{2}$

$Var[X] = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

$\therefore \sum_{i=1}^{20} X_i \xrightarrow{d} N\left(20 \times \frac{2}{3}, 20 \times \frac{1}{18}\right)$ by CLT

所求 $Pr[S \le 10] = Pr\left[\sum_{i=1}^{20} X_i \le 10\right]$

$\approx Pr\left[Z \le \frac{10 - \frac{40}{3}}{\sqrt{\frac{20}{18}}}\right] = \Phi\left[\frac{-\frac{10}{3}}{\sqrt{\frac{10}{9}}}\right] = \Phi(-\sqrt{10})$

$= 0.00078$

where $Z \sim N(0,1)$

※

CH5.

(17)

$p(|\bar{X}-\mu|<1) = p\left(\frac{|\bar{X}-\mu|}{\sigma/\sqrt{n}} < \frac{1}{5/\sqrt{n}}\right) = 0.95$, ∵ $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z$ by CLT, $Z \sim N(0,1)$

and $p\left(|Z| < \frac{\sqrt{n}}{5}\right) = 0.95$, $Z \sim N(0,1)$

∴ $1.96 \approx \frac{\sqrt{n}}{5} \Rightarrow n \approx 96.04$ n 取 96.

(18)

Let X_i : the weight of i-th company ship package with $i = 1, \dots, n$, $n=100$, $E(X_i) = 15$, $Var(X_i) = 100$.

∵ $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = 1500$, $Var\left(\sum_{i=1}^{n} X_i\right) \overset{\text{Indep}}{=\!=} \sum_{i=1}^{n} Var(X_i) = 100^2$.

∴ $p\left(\sum_{i=1}^{n} X_i > 1700\right) = p\left(\frac{\sum_{i=1}^{n} X_i - 1500}{\sqrt{100^2}} > \frac{1700-1500}{100}\right)$

$\approx p(Z > 2) = 0.0228$; where $Z \sim N(0,1)$

↑ by CLT

(23)

Let (X, Y) be the coordinates of the random point, then the joint pdf of (X,Y) is $f_{X,Y}(x,y) = \begin{cases} 1, & 0\le x\le 1, 0\le y\le 1 \\ 0, & \text{o.w.} \end{cases}$

Let $Z = I_A(X,Y) = \begin{cases} 1, & (X,Y) \in \text{object} \\ 0, & \text{o.w.} \end{cases}$

Then, $E(Z) = E(I_A(X,Y)) = p(Z=1) = p((X,Y) \in \text{object}) = \iint_{\text{object}} 1\,dx\,dy = A$.

Since $(X_i, Y_i) \overset{iid}{\sim} f_{X,Y}(x,y) = 1$, $0 \le x \le 1$, $0 \le y \le 1$

Let $Z_i = I_A(X_i, Y_i)$, then $Z_1, \dots, Z_n$ are i.i.d with mean A

By WLLN.

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{p} E(Z) = A \quad \text{as } n \to \infty$$

∴ We can use to estimate Area A.