CH2

21.

$$P(X > n+k-1|X > n-1) = \frac{P(X > n+k-1, X > n-1)}{P(X > n-1)} = \frac{P(X \geq n+k, X \geq n)}{P(X \geq n)}$$

$$= \frac{P(X \geq n+k)}{P(X \geq n)} = \frac{\sum_{i=n+k}^{\infty}(1-p)^{i-1}p}{\sum_{i=n}^{\infty}(1-p)^{i-1}p} = (1-p)^k = \frac{(1-p)^k p}{1-(1-p)}$$

$$= \sum_{i=k+1}^{\infty}(1-p)^{i-1}p = p(X \geq k+1) = P(X > k)$$

$X \sim Geo(p)$即表示 X 爲執行多次 independent bernoulli trials 直到一次成功所需要的次數, 而$\bar{F}(x) = P(X > x)$爲第 x 次後仍失敗的機率.
可以觀察出$P(X > n+k-1|X > n-1)$在給定$X > n-1$條件下, 表示在前 n-1次的 bernoulli trials 都爲失敗($Y_i = 0, Y_i \sim Ber(p), i = 1, 2, ..., n-1$)的前提下而從$X > n-1+k, k = 0, 1, 2, ...$的部分, 即$Y_{i=n}, Y_{i=n+1}, ..., Y_{i=n-1+k}, ...$, 所形成的分配與$P(X > k)$會相同, 皆服從$Geo(p)$, 因此可以看出 memoryless property.

31.

Def X be the number of phone calls in 10 mins.

(a) $\lambda = 2/hr, t = 10(mins) = 1/6(hr), \lambda t = 1/3$

$X \sim Poi(1/3), x = 0, 1, 2, ...$

$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1/3} \doteqdot 0.28$

(b) DefT be the time until the first phone call(mins).

$\lambda = 1/30$(per mins)

$T \sim Exp(1/30), t \geq 0$

$P(T \geq t) = 1 - (1 - e^{-t/30}) = e^{-t/30} \leq 0.5$

$t = 30ln2 \geq 20.79(mins)$

48.

Def. $T \sim Exp(\lambda), t > 0$

$P(T < 1) = 0.05 => 1 - e^{-\lambda} = 0.05$

$=> e^{-\lambda} = 0.95$

$=> \lambda = -ln(0.95) = 0.0513$

55.

$X \sim N(\mu, \sigma^2), Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$P(\mu - c \leq X \leq \mu + c) = P(\frac{-c}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{c}{\sigma}) = P(\frac{-c}{\sigma} \leq Z \leq \frac{c}{\sigma}) = 0.95$

$\therefore Z_{0.975} = 1.96 = \frac{c}{\sigma} => c = 1.96\sigma$

61.

$X \sim \Gamma(\alpha, \lambda), M_X(t) = (\frac{\lambda}{\lambda - t})^{\alpha}$

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Let $Y = cX, , M_Y(t) = E[e^{Yt}] = E[e^{cXt}] = (\frac{\lambda}{\lambda - ct})^\alpha = (\frac{\lambda/c}{\lambda/c - t})^\alpha$

$\therefore X \sim \Gamma(\alpha, \lambda), Y = cX \sim \Gamma(\alpha, \frac{\lambda}{c})$

CH3

22.

令X_1爲(t_0, t_1)發生事件的個數,X_2爲(t_1, t_2)發生事件的個數.

$X_1 \sim Poi(\lambda_1), X_2 \sim Poi(\lambda_2)$

$\because X_1 \amalg X_2 => f(x_1, x_2) = e^{-(\lambda_1+\lambda_2)} \frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{x_2}}{x_2!}, \quad x_1, x_2 = 0, 1, 2, ...$

令$N = X_1 + X_2, N \sim Poi(\lambda_1 + \lambda_2)$

$\therefore$所求爲$f(X_1|X_1 + X_2 = n) = \frac{f(X_1, X_1+X_2=n)}{f_N(n)}$

$$= \frac{f(X_1 = x_1, X_2 = n - x_1)}{f_N(n)} = \frac{\frac{\lambda_1^{x_1}}{x_1!} \frac{\lambda_2^{(n-x_1)}}{(n-x_1)!}}{\frac{(\lambda_1+\lambda_2)^n}{n!}} = \frac{\lambda_1^{x_1} \lambda_2^{n-x_1}}{(\lambda_1 + \lambda_2)^n} \frac{n!}{x_1!(n - x_1)!}$$

$$= \binom{n}{x_1} (\frac{\lambda_1}{\lambda_1 + \lambda_2})^{x_1} (\frac{\lambda_2}{\lambda_1 + \lambda_2})^{n-x_1}$$

$$=> X_1 \sim Bin(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}), x_1 = 0, 1, 2, ..., n$$

26.

$P \sim U(0, 1), 0 < p < 1 \quad X|P = p \sim Ber(p), x = 0, 1$

$f(x|P = p) = p^x(1 - p)^{1-x}, x = 0, 1$

$f(x, p) = f(x|P = p)f(p) = p^x(1 - p)^{1-x}, x = 0, 1 \quad , 0 < p < 1$

$\therefore f(p|X = x) = \frac{f(p, x)}{f_X(x)} = \frac{p^x(1 - p)^{1-x}}{\int_0^1 p^x(1 - p)^{1-x} dp}$

$= 2p^x(1 - p)^{1-x}, x = 0, 1 \quad 0 < p < 1$

其中2可以由以下公式得知$\int_0^1 x^{\alpha-1}(1 - x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$

$$\int_0^1 p^{(x+1)-1}(1 - p)^{(2-x)-1} dp = \frac{\Gamma(x + 1)\Gamma(2 - x)}{\Gamma(x + 1 + 2 - 2)} = \frac{1}{2}, x = 0, 1$$

CH4

89.

$X_i \sim N(\mu_i, \sigma_i^2), Y = \sum_{i=1}^n \alpha_i X_i$

$$M_Y(t) = E(e^{Yt}) = E(e^{\sum_{i=1}^n \alpha_i X_i t}) = \prod_{i=1}^n E(e^{\alpha_i X_i t}) = \prod_{i=1}^n M_{X_i}(\alpha_i t)$$

$$= \prod_{i=1}^n e^{\alpha_i \mu_i t + \frac{\alpha_i^2 \sigma_i^2}{2} t^2} = e^{t \sum \alpha_i \mu_i + t^2 \sum \frac{\alpha_i^2 \sigma_i^2}{2}}$$

$\therefore$ By the uniqueness of mgf, $Y \sim N(\sum_{i=1}^n \alpha_i \mu_i, \sum_{i=1}^n \alpha_i^2 \sigma_i^2)$

CH6

8.

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$$X, Y \sim Exp(1) \equiv \Gamma(\frac{2}{2}, 1)$$

$$2X, 2Y \sim \Gamma(\frac{2}{2}, \frac{1}{2}) \equiv \chi^2_2$$

$$\because F_{m,n} = \frac{\chi^2_m/m}{\chi^2_n/n}$$

$$\therefore \frac{X}{Y} = \frac{2X}{2Y} = \frac{\chi^2_2/2}{\chi^2_2/2} \sim F_{2,2}$$

11.

$$X_1, ..., X_n \sim N(\mu_x, \sigma^2_X = \sigma^2), Y_1, ..., Y_m \sim N(\mu_Y, \sigma^2_Y = \sigma^2)$$

$$\because \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{\sigma^2_X} = \frac{(n-1)S^2_X}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{\sum_{i=1}^m (Y_i - \overline{Y})^2}{\sigma^2_Y} = \frac{(m-1)S^2_Y}{\sigma^2} \sim \chi^2_{m-1}$$

$$\frac{S^2_X}{S^2_Y} = \frac{\frac{(n-1)S^2_X}{\sigma^2}/n-1}{\frac{(m-1)S^2_Y}{\sigma^2}/m-1} \sim F_{n-1,m-1}$$

$p(F_{n-1,m-1} > c)$可查 F 分配表得知