Ch 4

42.

(i) $X \sim \text{Exp}(\lambda)$, $0 < x < \infty$, $\lambda > 0$

$E[X] = \frac{1}{\lambda}$, $Var[X] = \frac{1}{\lambda^2}$, $\sigma = \frac{1}{\lambda}$

$$\begin{aligned}
P[|X - E[X]| > k\sigma] &= P\left[\left|X - \frac{1}{\lambda}\right| > \frac{k}{\lambda}\right] \\
&= P\left[X - \frac{1}{\lambda} > \frac{k}{\lambda},\ X - \frac{1}{\lambda} < -\frac{k}{\lambda}\right] \\
&= P\left[X - \frac{1}{\lambda} > \frac{k}{\lambda}\right] + P\left[X - \frac{1}{\lambda} < -\frac{k}{\lambda}\right] \\
&= 1 - e^{-\lambda x}\Big|_{\frac{1+k}{\lambda}}^{\infty} + \boxed{1 - e^{-\lambda x}\Big|_{0}^{\frac{1-k}{\lambda}}} \\
&= e^{-1-k} + \boxed{1 - e^{-1+k}} = \begin{cases} 0.05, & k = 2 \\ 0.018, & k = 3 \\ 0.007, & k = 4 \end{cases}
\end{aligned}$$

$\Rightarrow$ $1-k>0$ 時, 才有 prob $\because 0 < x < \infty$

(ii) Chebyshev's inequality: $P[|X - E[X]| \geq a] \leq \frac{\sigma^2}{a^2}$ ※

令 $a = \frac{k}{\lambda}$, 則 $P\left[|X - E[X]| \geq \frac{k}{\lambda}\right] \leq \frac{1}{k^2} = \begin{cases} \frac{1}{4} > 0.05, & k = 2 \\ \frac{1}{9} > 0.018, & k = 3 \\ \frac{1}{16} > 0.007, & k = 4 \end{cases}$

※

49.

(a) $E[Z] = E[\alpha X + (1-\alpha) Y] = \alpha E[X] + (1-\alpha) E[Y]$

$= \alpha u + (1-\alpha) u = u$ ※

(b) $Var[Z] = \alpha^2 Var[X] + (1-\alpha)^2 Var[Y] = \alpha^2 \sigma_X^2 + (1-\alpha)^2 \sigma_Y^2$

$$\begin{cases} \dfrac{\partial Var[Z]}{\partial \alpha} = 2\alpha\sigma_X^2 - 2(1-\alpha)\sigma_Y^2 = 0, \text{ when } \alpha = \dfrac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2} \\ \dfrac{\partial^2 Var[Z]}{\partial \alpha^2} = 2\sigma_X^2 + 2\sigma_Y^2 > 0 \end{cases}$$

$\therefore$ 當 $\alpha = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2}$ 時, $Var[Z] = \frac{\sigma_X^2 \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$ 為最小值

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(c)

$$Var\left(\frac{X+Y}{2}\right) = \frac{\sigma_X^2+\sigma_Y^2}{4}$$

We prefer to use the average $Z = \frac{X+Y}{2}$,

if it has the smaller variation than that either of X or Y alone

$$\begin{matrix} Var[Z] < Var[X] \\ Var[Z] < Var[Y] \end{matrix} \Rightarrow \begin{cases} \frac{\sigma_X^2+\sigma_Y^2}{4} < \sigma_X^2 \\ \frac{\sigma_X^2+\sigma_Y^2}{4} < \sigma_Y^2 \end{cases} \Rightarrow \begin{cases} \frac{\sigma_X^2}{\sigma_Y^2} > \frac{1}{3} \\ \frac{\sigma_X^2}{\sigma_Y^2} < 3 \end{cases} \Rightarrow \frac{1}{3} < \frac{\sigma_X^2}{\sigma_Y^2} < 3$$

※

59.

$$f(x,y) = \frac{1}{\pi}, \quad 0 < x^2+y^2 < 1$$

$$f(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx = \frac{2\sqrt{1-y^2}}{\pi}, \quad -1 < y < 1$$

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2\sqrt{1-x^2}}{\pi}, \quad -1 < x < 1$$

$$COV(X,Y) = E[XY] - E[X]\cdot E[Y] = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{xy}{\pi}\, dx\, dy - \int_{-1}^{1} x\cdot\frac{2\sqrt{1-x^2}}{\pi}\, dx \cdot \int_{-1}^{1} y\cdot\frac{2\sqrt{1-y^2}}{\pi}\, dy = 0$$

但 $f(x,y) \neq f(x)\cdot f(y)$, $\therefore X \not\perp\!\!\!\perp Y$

61.

(a) $Cov(X,Y) = E[XY] - E[X]\cdot E[Y]$

$$= \int_0^1\int_0^y x\cdot y\cdot 2\,dx\,dy - \int_0^1\int_0^y x\cdot 2\,dx\,dy \times \int_0^1\int_0^y y\cdot 2\,dx\,dy = \frac{1}{36}$$

$$Var[X] = \int_0^1\int_0^y x^2 f(x,y)\,dx\,dy - \left(\frac{1}{3}\right)^2 = \frac{3}{54}$$

$$Var[Y] = \int_0^1\int_0^y y^2 f(x,y)\,dx\,dy - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

$$\therefore\ \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{1}{2} > 0$$，為正相關

∴ 代表 Y 小時，X 必定小
X 大時，Y 必定大 ※

(b)

$$E[X|Y=y] = \int_0^y x\cdot f(x|y)\,dx = \int_0^y x\frac{f(x,y)}{f(x,y)dx}dx = \int_0^y x\cdot\frac{2}{2y}dx = \frac{y}{2}$$

$$E[Y|X=x] = \int_x^1 y\cdot f(y|x)\,dy = \int_x^1 y\cdot\frac{f(x,y)}{\int_x^1 f(x,y)dy}dy = \int_x^1 y\frac{2}{2(1-x)}dy = \frac{1+x}{2}$$

intuition: ∵ $X < Y$, $X|Y=y \sim U(0,y)$, $Y|X=x \sim U(x,1)$ ※

(c)

$E[X|Y] = \frac{Y}{2}$，令 $Z = \frac{Y}{2}$

$$\therefore P[Z<z] = P[Y \le 2z] = \int_0^{2z} f(y)\,dy = \int_0^{2z} 2y\,dy$$

$$\therefore f_Z(z) = 8z,\quad 0<z<\frac{1}{2}$$ ※

$E[Y|X] = \frac{1+X}{2}$，令 $W = \frac{1+X}{2}$

$$\therefore P[W<w] = P[X \le 2w-1] = \int_0^{2w-1} f(x)\,dx = \int_0^{2w-1}\int_x^1 f(x,y)\,dy\,dx = \underbrace{}_{\int_0^{2w-1} 2(1-x)dx}$$

$$\therefore f_W(w) = 8(1-w),\quad \frac{1}{2} \le w \le 1$$ ※

(d)

請由老師講義58頁的定理來做

$$a = u_Y - b\,u_X \quad , \quad b = \rho\frac{\sigma_Y}{\sigma_X}.$$

$$f_X(x) = \int_x^1 2\,dy = 2-2x \;, \quad 0<x<1$$

$$f_Y(y) = \int_0^y 2\,dx = 2y \;, \quad 0<y<1$$

$$u_X = E[X] = \int_0^1 2x - 2x^2\,dx = 1-\frac{2}{3} = \frac{1}{3} \;, \quad E[X^2] = \int_0^1 2x^2 - 2x^3\,dx = \frac{1}{6}$$

$$u_Y = E[Y] = \int_0^1 2y^2\,dy = \frac{2}{3} \;, \quad E[Y^2] = \int_0^1 2y^3\,dy = \frac{1}{2}$$

$$\sqrt{Var[X]} = \sqrt{E[X^2] - E[X]^2} = \sqrt{\frac{1}{18}} = \sigma_X$$

$$\sqrt{Var[Y]} = \sqrt{E[Y^2] - E[Y]^2} = \sqrt{\frac{1}{18}} = \sigma_Y$$

$$\therefore \begin{cases} a = \frac{2}{3} - \frac{1}{3}b \\ b = \frac{1}{2}\cdot 1 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\ b = \frac{1}{2} \end{cases} \Rightarrow Y = \frac{1}{2} + \frac{1}{2}X \quad ※$$

而最小MSE為 $\sigma_Y^2(1-\rho^2) = \frac{1}{18}\cdot\left(1-\frac{1}{4}\right) = \frac{1}{24}$ ※

(e) 由(d)知, best predictor $h(x) = \frac{1}{2} + \frac{1}{2}X = E[Y|X]$

minimal $MSE = E(Var[Y|X]) = \frac{1}{24}$ ※

$$\left(\begin{aligned} E[V(Y|X)] &= V(Y) - V(E[Y|X]) \\ &= \frac{1}{18} - V\left(\frac{1+X}{2}\right) = \frac{1}{18} - \frac{1}{4}\times\frac{1}{18} = \frac{1}{24} \end{aligned} \right)$$

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67.

circumferent $= 2X + 2Y$, area $= XY$.

$X \sim U(0,1)$

$Y|X \sim U(0, X)$

$\rightarrow E(2X+2Y) = 2E(X+Y) = 2E[E(X+Y|X)] = 2E(X+\frac{X}{2}) = E(3X)$

$= \frac{3}{2}$

$\rightarrow E(XY) = E[E(XY|X)] = E(\frac{X^2}{2}) = \frac{1}{6}$

75. Let $T \sim Exp(\frac{1}{\lambda})$

$U|T \sim U(0,T)$

$E(U) = E[E(U|T)] = E(\frac{T}{2}) = \frac{1}{2\lambda}$.

$Var(U) = Var(E(U|T)) + E[Var(U|T)]$

$= Var(\frac{T}{2}) + E[\frac{T^2}{12}]$

$= \frac{1}{4}\frac{1}{\lambda^2} + \frac{1}{12}\frac{2}{\lambda^2} = \frac{5}{12}\frac{1}{\lambda^2}$

76.

$$f(x,y)=\begin{cases}\frac{2}{\pi}, & x^2+y^2\le 1,\ y\ge 0.\\ 0, & \text{o.w}\end{cases}$$

$$f(x)=\int_0^{\sqrt{1-x^2}}\frac{2}{\pi}\,dy=\frac{2}{\pi}\sqrt{1-x^2}, \quad -1\le x\le 1$$

$$f(y)=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{2}{\pi}\,dx=\frac{4}{\pi}\sqrt{1-y^2}, \quad 0\le y\le 1.$$

$$f_{Y|X}(y)=\frac{f(x,y)}{f(x)}=\frac{1}{\sqrt{1-x^2}} \qquad 0\le y\le\sqrt{1-x^2}$$

$$f_{X|Y}(x)=\frac{f(x,y)}{f(y)}=\frac{1}{2\sqrt{1-y^2}} \qquad -\sqrt{1-y^2}\le x\le\sqrt{1-y^2}$$

① X be observed, the best prediction of $Y=E[Y|X]$.

$$E[Y|X]=\int_0^{\sqrt{1-x^2}}\frac{y}{\sqrt{1-x^2}}\,dy=\frac{\sqrt{1-x^2}}{2}$$

② Y be observed, the best prediction of $X=E[X|Y]$.

$$E[X|Y]=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{x}{2\sqrt{1-y^2}}\,dx=0$$

Ch4.

77.

$$f(x,y) = e^{-y} \quad , \quad 0 \le x \le y .$$

$$f(x) = \int_x^\infty e^{-y}\, dy = -e^{-y}\Big|_x^\infty = e^{-x} \quad , \quad 0 \le x \le \infty$$

$$f(y) = \int_0^y e^{-y}\, dx = y e^{-y} \qquad 0 \le y \le \infty$$

由 pdf → $X \sim Exp(1)$, $Y \sim Gamma(2,1)$.

(a)
$$E(XY) = \int_0^\infty \int_0^y xy e^{-y}\, dx\, dy = \int_0^\infty \frac{y^3}{2} e^{-y}\, dy .$$

$$= -\frac{y^3}{2} e^{-y}\Big|_0^\infty + \frac{3}{2}\int_0^\infty y^2 e^{-y}\, dy = \frac{3}{2} \times 2 = 3 .$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 3 - 1\times 2 = 1 .$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{1}{\sqrt{1\times 2}} = \frac{1}{\sqrt{2}} .$$

(b)
$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y} \qquad 0 \le x \le y .$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{e^{-y}}{e^{-x}} = e^{-y+x} \qquad x \le y \le \infty .$$

$$E(X|Y=y) = \int_0^y \frac{x}{y}\, dx = \frac{y}{2} ,$$

$$E(Y|X=x) = \int_x^\infty e^{-y+x} y\, dy = e^x\left[-ye^{-y}\Big|_x^\infty + \int_x^\infty e^{-y} dy \right] = 1+x .$$

(c) Let $T = E(X|Y) = \frac{Y}{2}$ $\therefore$ $Y = 2T$. $J = |2|$.

$$f_T(t) = 4te^{-2t} \qquad 0 \le t \le \infty .$$

Let $S = E(Y|X) = 1+X$ $\therefore$ $X = 1-S$ $J = 1$.

$$f_S(s) = e^{1-s} \quad , \quad 1 \le s \le \infty .$$

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80. $X \sim f(x) = 2x$, $0 < x < 1$.

① $M_X(t) = E(e^{xt}) = \int_0^1 e^{xt}(2x)\,dx = 2\int_0^1 xe^{xt}\,dx$

$$\stackrel{\text{IBP}}{=} 2\left(\frac{xe^{xt}}{t} - \int_0^1 \frac{e^{xt}}{t}\,dx\right) = 2\left((e^t - 0) - \frac{1}{t^2}e^{xt}\Big|_{x=0}^{x=1}\right)$$

$$= 2\left(\frac{te^t - e^t + 1}{t^2}\right) \#$$

② $E(X) = M'(t)\big|_{t=0}$, $M'(t) = \dfrac{2(t^2e^t - 2te^t + 2e^t - t)}{t^3}$, $\lim_{t\to0} M'(t) \stackrel{\text{L'Hospital}}{=} \lim_{t\to0} \dfrac{2e^t + te^t}{3} = \dfrac{2}{3} \#$

By def

$E(X) = \int_0^1 x(2x)\,dx = 2 \times \frac{x^3}{3}\Big|_0^1 = \frac{2}{3} \#$

③ $E(X^2) = M''(t)\big|_{t=0}$. $M''(t) = \dfrac{2(t^3e^t - 3t^2e^t + 6te^t - 6e^t + 2t)}{t^4}$, $\lim_{t\to0} M''(t) \stackrel{\text{LH}}{=} \dfrac{e^t(3+t)}{6} = \dfrac{1}{2} \#$

By def

$E(X^2) = \int_0^1 x^2(2x)\,dx = 2 \times \frac{x^4}{4}\Big|_0^1 = \frac{1}{2} \#$

91. $X \sim \text{Exp}(\lambda)$ $M_X(t) = \dfrac{\lambda}{\lambda - t}$, $0 < x < \infty$, $c > 0$, $\lambda > 0$.

$E(e^{cXt}) = \dfrac{\lambda}{\lambda - ct} = \dfrac{\frac{\lambda}{c}}{\frac{\lambda}{c} - t}$

$\because$ By MGF 唯一性. $\therefore cX \sim \text{Exp}\left(\frac{\lambda}{c}\right)$

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$\Theta \sim \Gamma(\alpha, \lambda)$. $M_{\Theta}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$, $t < \lambda$

$X|\Theta \sim Poi(\Theta)$. $M_{X|\Theta}(t) = e^{\Theta(e^t - 1)}$, $t \in \mathbb{R}$.

$$M_X(t) = E(e^{Xt}) = EE(e^{Xt}|\Theta) = E\left(e^{\Theta(e^t-1)}\right)$$

$$= \left(\frac{\lambda}{\lambda-(e^t-1)}\right)^{\alpha} = \left(\frac{\lambda}{1+\lambda-e^t}\right)^{\alpha} = \left(\frac{\frac{\lambda}{1+\lambda}}{1-\frac{1}{1+\lambda}e^t}\right)^{\alpha} = \left(\frac{\frac{\lambda}{1+\lambda}}{1-(1-\frac{\lambda}{1+\lambda})e^t}\right)^{\alpha}$$

By MGF 唯一性.

$$\alpha + X \sim NB\left(\alpha, \frac{\lambda}{1+\lambda}\right).$$

99. By Taylor expansion, we can get the approximation

$$Y = g(X) \approx g(a) + g'(a)(X-a) + \frac{g''(a)}{2}(X-a)^2.$$

(a). $g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $g''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$E(g(X)) \approx \sqrt{\mu} - \frac{1}{8}\sigma^2\mu^{-\frac{3}{2}}$

$Var(g(X)) \approx \frac{1}{4}\sigma^2\mu^{-1}$.

(b) $g'(x) = \frac{1}{x}$. $g''(x) = \frac{-1}{x^2}$

$E(g(X)) \approx \log\mu - \frac{1}{2}\sigma^2\mu^{-2}$

$Var(g(X)) \approx \sigma^2\mu^{-2}$

(c) $g'(x) = \frac{1}{\sqrt{1-x^2}}$ $g''(x) = x(1-x^2)^{-\frac{3}{2}}$

$E(g(X)) \approx \sin^{-1}\mu + \frac{1}{2}\mu(1-\mu^2)^{-\frac{3}{2}}\sigma^2$

$Var(g(X)) \approx \frac{\sigma^2}{1-\mu^2}$