Ch 2

67.

(a) $f_X(x) = \frac{dF(x)}{dx} = \frac{\beta}{\alpha^\beta} \cdot x^{\beta-1} \cdot e^{-\left(\frac{x}{\alpha}\right)^\beta}$, $x \geq 0$, $\alpha > 0$, $\beta > 0$

※

(b) $W \sim$ weibull (α, β).

$f_W(w) = \frac{\beta}{\alpha^\beta} \cdot w^{\beta-1} \cdot e^{-\left(\frac{w}{\alpha}\right)^\beta}$, $w \geq 0$, $\alpha > 0$, $\beta > 0$

$X = \left(\frac{W}{\alpha}\right)^\beta \Rightarrow W = \alpha \cdot X^{\frac{1}{\beta}}$

$|J| = \left|\frac{dw}{dx}\right| = \frac{\alpha}{\beta} \cdot X^{\frac{1-\beta}{\beta}}$

$\therefore f_X(x) = \cancel{\frac{\beta}{\alpha^\beta}} \cdot \left(\cancel{\alpha \cdot x^{\frac{1}{\beta}}}\right)^{\beta-1} \cdot e^{-\left(\frac{\alpha \cdot x^{\frac{1}{\beta}}}{\alpha}\right)^\beta} \cdot \cancel{\frac{\alpha}{\beta}} \cdot \cancel{x^{\frac{1-\beta}{\beta}}} = e^{-x}$

$\sim EXP(1)$, $x \geq 0$

※

(c)

$U_i \overset{iid}{\sim} Unif(0,1)$, $i = 1, 2, \cdots, n$

共 n 個 $unif(0,1)$ 的生成值

令 $X \sim F_X(x) = Weibull(\alpha, \beta)$

則 $F_X(X) \sim Unif(0,1)$

$\Rightarrow X \sim F_X^{-1}(U) = \alpha\left[-\ln(1-U)\right]^{\frac{1}{\beta}}$

故 $F_X^{-1}(U_i) = x_i$, $i = 1, 2, \cdots, n$

$\therefore$ 此 n 個 x_i 值即為 Weibull 分配的生成值

※

Ch 2

10. 令 R 代表圓半徑

$R \sim Exp(\lambda)$, $f_R(r) = \lambda e^{-\lambda r}$, $0 < r < \infty$

令 $X = \pi R^2 \Rightarrow R = \left(\frac{X}{\pi}\right)^{\frac{1}{2}}$, $|J| = \left|\frac{dR}{dX}\right| = \frac{1}{2\sqrt{\pi}} \cdot X^{-\frac{1}{2}}$

$\therefore f_X(x) = \lambda e^{-\lambda\left(\frac{x}{\pi}\right)^{\frac{1}{2}}} \cdot \frac{1}{2\sqrt{\pi}} \cdot x^{-\frac{1}{2}} = \frac{\lambda}{2\sqrt{\pi}} \cdot x^{-\frac{1}{2}} \cdot e^{-\lambda\left(\frac{x}{\pi}\right)^{\frac{1}{2}}}$, $0 < x < \infty$ ※

Ch 3

10. $f(x, y) = x e^{-x(y+1)}$, $0 \le x < \infty$, $0 \le y < \infty$

(a) $f_X(x) = \int_0^\infty x e^{-x(y+1)} dy = e^{-x}$, $0 < x < \infty$

$f_Y(y) = \int_0^\infty x e^{-x(y+1)} dx = (y+1)^{-2}$, $0 < y < \infty$

$\therefore f(x, y) \ne f_X(x) \cdot f_Y(y)$, $\therefore X \not\perp\!\!\!\perp Y$

(b) $f(x|y) = \frac{f(x,y)}{f_Y(y)} = x(y+1)^2 \cdot e^{-x(y+1)}$, $0 \le x < \infty$, $0 \le y < \infty$

$f(y|x) = \frac{f(x,y)}{f_X(x)} = x e^{-xy}$, $0 \le y \le \infty$, $0 \le x < \infty$

16. $X_1 \sim U(0,1)$, $0 < x_1 < 1$, $X_2/x_1 \sim U(0, x_1)$

$f_{X_1}(x_1) = 1$, $0 \le x_1 \le 1$, $f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1}$, $0 \le x_2 \le x_1$

$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1}$, $0 \le x_2 \le x_1 \le 1$

$f_{X_2}(x_2) = \int_{x_2}^1 \frac{1}{x_1} dx_1 = \ln x_1 \Big|_{x_2}^1 = -\ln x_2$, $0 \le x_2 \le 1$

※

Ch3

21.

Let $G(y) = P(Y \le y \mid \text{The soil has been determined to present})$

$\rightarrow G(y) = P(Y \le y)$

$\therefore g(y) = \frac{d}{dy} G(y)$

$\because G(y) = \dfrac{\int_0^y R(t) f(t)\, dt}{\int_0^\infty R(x) f(x)\, dx}$ → $\{x \le y$, 且可被 detect$\}$ (numerator); → 可被 detect (denominator)

$$\Rightarrow g(y) = \frac{d}{dy} G(y) = \frac{R(y) f(y)}{\int_0^\infty R(x) f(x) dx}.$$

25.

Let $\omega = \begin{cases} 1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2} \end{cases}$, $\omega \perp x$. $f_\omega(\omega) = \frac{1}{2}$, $\omega = -1, 1$.

then $Y = \omega X$.

set $z = \omega$, $X = \frac{Y}{z}$, $\omega = z$. $|J| = |\frac{1}{z}|$

$f_{Y,Z}(y, z) = f_{X,\omega}(\frac{y}{z}, z) |\frac{1}{z}| = f_X(\frac{y}{z}) f_\omega(z) |\frac{1}{z}|$

$f_Y(y) = \sum_{z=-1,1} f_X(\frac{y}{z}) f_\omega(z) |\frac{1}{z}| = \frac{1}{2} f_X(y) + \frac{1}{2} f_Y(-y)$

$\rightarrow f_Y(y) = f_Y(-y)$

44.

$X = \begin{cases} 0 & p=\frac{1}{3} \\ 1 & p=\frac{1}{3} \\ 2 & p=\frac{1}{3} \end{cases}$ $\qquad Y = \begin{cases} 0 & p=\frac{1}{3} \\ 1 & p=\frac{1}{3} \\ 2 & p=\frac{1}{3} \end{cases}$

X+Y joint:

X \ Y	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

$\rightarrow \quad X+Y = \begin{cases} 0 & P=\frac{1}{9} \\ 1 & P=\frac{2}{9} \\ 2 & P=\frac{1}{3} \\ 3 & P=\frac{2}{9} \\ 4 & P=\frac{1}{9} \end{cases}$

64.

Let $U=X+Y$, $V=\frac{X}{Y}$, $\begin{matrix} u\geq 0 \\ v\geq 0 \end{matrix}$ and range of u,v are irrelent.

$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$

$x=\frac{uv}{v+1}$ $\qquad |J| = \frac{vu+u}{(v+1)^2}$

$y=\frac{u}{v+1}$

$\therefore f_{u,v}(u,v) = f_{X,Y}\left(\frac{uv}{v+1}, \frac{u}{v+1}\right)|J| = \frac{\lambda^2 u}{(v+1)^2} e^{-\lambda u}$.

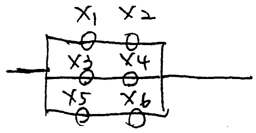
$\Rightarrow f_{u,v}(u,v) = \frac{1}{(v+1)^2}\lambda^2 u e^{-\lambda u} = g(u)\cdot h(v)$

$\therefore u \perp\!\!\!\perp v \Rightarrow X+Y \perp\!\!\!\perp \frac{X}{Y}$.

CH3. 66.

X_i: one component's life time, $i=1,2,3,4,5,6$.

$X_i \sim Exp(\lambda)$



$Y = \max\{\min(X_1,X_2), \min(X_3,X_4), \min(X_5,X_6)\}$

Y: the system's lifetime.

其中 令 $W_1=\min(X_1,X_2)$, $W_2=\min(X_3,X_4)$, $W_3=\min(X_5,X_6)$

$$
\begin{aligned}
P(W_1\le w_1) &= 1-P(\min(X_1,X_2) > w_1) \\
&= 1-(1-F_X(w_1))^2 \\
&= 1-e^{-2\lambda w_1}, \quad w_1 \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
\therefore P(Y\le y) &= P(\max\{W_1,W_2,W_3\}\le y) \\
&= P(W_1\le y)P(W_2\le y)P(W_3\le y) \\
&= (1-e^{-2\lambda y})^3, \quad y\ge 0.
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) = \frac{d}{dy}F_Y(y) &= 3(1-e^{-2\lambda y})^2\cdot 2\lambda e^{-2\lambda y} \\
&= 6\lambda e^{-2\lambda y}(1-e^{-2\lambda y})^2, \quad y\ge 0
\end{aligned}
$$

20.

$$F(x,y) = P(Y_1 \le x, Y_n \le y)$$
$$= P(Y_n \le y) - P(Y_1 > x, Y_n \le y)$$
$$= P(\max\{X_1, \dots, X_n\} \le y) - P(x < X_i \le y, i=1,\dots,n)$$
$$= (F(y))^n - (F(y) - F(x))^n, \quad x \le y$$

CH4.

18. $U_{(n)} \sim f_n(u)$, $U_{(1)} \sim f_1(u)$

$f_n(u) = n\, u^{n-1}$, $0 \le u \le 1$, $U_{(n)} \sim Beta(n, 1)$

$f_1(u) = n(1-u)^{n-1}$, $0 \le u \le 1$, $U_{(1)} \sim Beta(1, n)$

$E(U_{(n)}) = \frac{n}{n+1}$. $E(U_{(1)}) = \frac{1}{n+1}$

$E(U_{(n)} - U_{(1)}) = \frac{n-1}{n+1}$

36.

(a) $X \sim U(0,1)$ $f(x) = 1$.

$Y = \sqrt{X}$, $f(y) = f(x(y)) \left| \frac{dx}{dy} \right| = 2y$.

$E(Y) = \int_0^1 y \times 2y \, dy = \frac{2}{3}$

(b) $\because Y = g(x) = \sqrt{x}$.