HW13.CH9

12.
$X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \theta e^{-\theta x}$
$\Rightarrow L(\theta|\underset{\sim}{X}) = \theta^n e^{-\theta \sum X_i}$, $0 < \theta < \infty$
$\Rightarrow l(\theta) = \ln L(\theta|\underset{\sim}{X}) = n \ln \theta - \theta \sum X_i$
Let $\frac{\partial l(\theta)}{\partial \theta} = 0$, $\hat{\theta} = \frac{1}{\bar{X}}$, $l^{''}(\hat{\theta}) < 0$, $\therefore \hat{\theta}$ is MLE of θ
$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
$\therefore \Omega = \{\theta : \theta \in R^+\}$, $\Omega_0 = \{\theta_0\}$
$\Rightarrow \Lambda = \frac{\theta_0^n exp(-\theta_0 \sum X_i)}{(\bar{X})^{-n} exp(-\frac{1}{\bar{X}} \sum X_i)} = \theta_0^n exp(-n\theta_0 \bar{X})(\frac{1}{\bar{X}})^{-n} exp(n) = \theta_0^n e^n [\bar{X} exp(-\theta_0 \bar{X})]^n$
$\therefore \Lambda \leq C^* \Leftrightarrow \bar{X} exp(-\theta_0 \bar{X}) \leq (\frac{C^*}{\theta_0^n e^n})^{\frac{1}{n}} = c$
$\therefore$ The reject region $\bar{X} exp(-\theta_0 \bar{X}) \leq c$

13.(a)
form problem.12, the reject region is $\bar{X} exp(-\bar{X}) \leq c$
Let $g(y) = ye^{-y} \Rightarrow g^{'}(y) = e^{-y}(1-y)$
$$\therefore g^{'}(y) \begin{cases} = 0, & y = 1 \\ > 0, & y < 1 \\ < 0, & y > 1 \end{cases}$$
So g is concave and $g(y) \leq$ c iff $y \geq x_1$ or $y \leq x_0$
$\therefore \{\bar{X} exp(-\bar{X}) \leq c\} = \{\bar{X} \leq x_0\} \bigcup \{\bar{X} \geq x_1\}$, where x_0,x_1 are determined by c

13.(b)
$\therefore \alpha = 0.05$ and α =p(reject $H_0|\theta_0 = 1$)
$\therefore$ We should choose c, so $P(\bar{X} exp(-\bar{X}) \leq c|\theta_0 = 1) = 0.05$

13.(c)
When $\theta_0 = 1$, $X_i \overset{iid}{\sim} f(x) = e^{-x} \equiv \Gamma(1,1)$
$\therefore \sum_{i=1}^{10} X_i \sim \Gamma(10,1)$
$Y = \bar{X} \sim \Gamma(10, \frac{1}{10})$, so we find c that satisfy $\int_A \frac{10^{10}}{\Gamma(10)} y^9 e^{-10y} dy$
$A = \{ye^{-y} \leq c\}$

26.
a. T
b. F, p-value=0.03 > 0.02 =α, so not reject H_0.
c. T
d. F, p-value is probability that the smallest singnificant level to reject H_0.
e. F, by TextBook p.343, example A $\Rightarrow$ p-value=0.86,
but the likelihood ratio=$exp(\frac{0.0319}{-2}) = 0.9872$
f. F, $\because \chi_4^2(0.95) = 9.488 > 8.5$
$\therefore P_r(\chi_4^2 > 9.488) = 0.05 < P_r(\chi_4^2 > 8.5) = 0.075$

40.

Jointly made by 趙致平, 許家綸, 黃彥霖 助教

X_1, X_2 has multinomial distribution with $X_1 + X_2 = n, p_1 + p_2 = 1$

$$\sum_{i=1}^{2} \frac{(X_i - np_i)^2}{np_i} = \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_2 - np_2)^2}{np_2} = \frac{(X_1 - np_1)^2}{np_1} + \frac{(n - X_1 - n(1-p_1))^2}{n(1-p_1)} = \frac{(X_1 - np_1)^2}{np_1(1-p_1)}$$

41.

$H_0 : p_1 = p_2 = \cdots = p_m = p$

$\because X_i \sim Bin(n_i, p_i),\ p(x_i) = \begin{pmatrix} n_i \\ x_i \end{pmatrix} p_i^{x_i}(1-p_i)^{n_i - x_i} = \begin{pmatrix} n_i \\ x_i \end{pmatrix} p_{ij}^{x_{ij}}, j = 1, 2$

where $\begin{cases} p_{i1} = & p_i \\ p_{i2} = & 1 - p_i \\ x_{i1} = & x_i \\ x_{i2} = & n_i - x_i \end{cases}$

$$\therefore \lambda(\underset{\sim}{X}) = \frac{\hat{p}^{\sum x_i}(1-\hat{p})^{\sum n_i - \sum x_i}}{\prod_{i=1}^{m} \hat{p}_i^{x_i}(1-\hat{p}_i)^{n_i - x_i}} = \prod_{i=1}^{m}\prod_{j=1}^{2}(\frac{\hat{p}_j}{\hat{p}_{ij}})^{x_{ij}},$$

where $\begin{cases} \hat{p}_{j=1} = & \hat{p} & = & \sum X_i / \sum n_i \\ \hat{p}_{j=2} = & 1 - \hat{p} & = & 1 - \sum X_i / \sum n_i \\ \hat{p}_{i1} = & \hat{p}_i & = & X_i / n_i \\ \hat{p}_{i2} = & 1 - \hat{p}_i & = & 1 - X_i / n_i \end{cases}$

由課本 p.342, 可以得到

$$-2log\lambda(\underset{\sim}{X}) = 2\sum_{i=1}^{m}\sum_{j=1}^{2} n_i \hat{p}_{ij} log \frac{\hat{p}_{ij}}{\hat{p}_j}$$

and under H_0 by lager sample distribution we can get

$$-2log\Lambda \approx \sum_{i=1}^{m}\sum_{j=1}^{2} \frac{(X_{ij} - n_i\hat{p})^2}{n_i p} = \sum_{i=1}^{m} \frac{(X_i - n_i\hat{p})^2}{n_i\hat{p}(1-\hat{p})} \overset{d}{\sim} \chi^2_{m-1}$$

and we can determine the rejection region with size α

$$\sum_{i=1}^{m} \frac{(X_i - n_i\hat{p})^2}{n_i\hat{p}(1-\hat{p})} > \chi^2_{m-1}(\alpha)$$

The χ^2 distribution with $m-1$ degrees of freedom.

Since $dim(\Omega) = m$ $(p_1, ..., p_m)$ and $dim(\Omega_0) = 1$ $(p_1 = ... = p_m)$.

43.(a)

$H_0 : p = 1/2 \Leftrightarrow H_0$:p_1=P(Head)=p_2=P(Tail)=1/2

$x_2 = 9207, x_2 = 8743, p_1 = p_2 = \frac{1}{2}, n = 17950$

$\sum_{i=1}^{2} \frac{(x_i - np_i)^2}{np_i} = 11.99 > \chi^2_1(0.95) = 3.841459$

df=1, since $dim(\Omega) = 1$ (p_1, p_2) and $dim(\Omega_0) = 0$ $(p_1 = p_2 = \frac{1}{2})$

reject $H_0 : p = 1/2$

43.(b)

Let $q_1, ..., q_5$ means P(i-th coin head)=$q_i, i = 1, ...5$

$H_0 : q_1 = q_2 = ... = q_5 \equiv q = \frac{1}{2}$ with p_k=P(5 coins k heads)=$C_k^5 q^k (1-q)^{5-k}$

p_0 =p(0 heads)=$\frac{1}{32}$, $E_0 = 3590p_0 = 112.1875$

p_1 =p(1 heads)=$\frac{5}{32}$, $E_1 = 3590p_1 = 560.9375$

p_2 =p(2 heads)=$\frac{10}{32}$, $E_2 = 3590p_2 = 1121.875$

p_3 =p(3 heads)=$\frac{10}{32}$, $E_3 = 3590p_3 = 1121.875$

p_4 =p(4 heads)=$\frac{5}{32}$, $E_4 = 3590p_4 = 560.9375$

p_5 =p(5 heads)=$\frac{1}{32}$, $E_5 = 3590p_5 = 112.1875$

$$\sum_{i=0}^{5} \frac{(O_i - E_i)^2}{E_i} = 21.56813 > \chi_5^2(0.95) = 11.0705$$

df=5, since $dim(\Omega) = 5$ $(p_0, ..., p_5, \sum_{i=0}^{5} p_i = 1)$ and

$dim(\Omega_0) = 0$ $(q_1 = ... = q_5 = \frac{1}{2})$

reject H_0 : all coins are fair

43.(c)

$H_0 : q_1 = q_2 = ... = q_5 \equiv q$

under H_0, $L(q|\underset{\sim}{X}) = \prod_{i=1}^{3590} C_{x_i}^5 q^{x_i}(1-q)^{5-x_i}$, $\hat{q} = \frac{\sum X_i}{3590\times 5} = 0.5129248$

同 (b) 小題作法, 但 $q = \hat{q}$

$$\sum_{i=0}^{5} \frac{(O_i - E_i)^2}{E_i} = 8.743702 < \chi_4^2(0.95) = 9.487729$$

df=4, since $dim(\Omega) = 5$ $(p_0, ..., p_5, \sum_{i=0}^{5} p_i = 1)$ and

$dim(\Omega_0) = 1$ $(q_1 = ... = q_5 \equiv q)$

Hence we do not reject H_0.