HW10

1.(a)

$T = \dfrac{Y}{\theta} => \theta T = Y, J = |\dfrac{dy}{dT}| = \theta$

$f_T(t) = \dfrac{2(\theta - \theta t)}{\theta^2} = 2(1-t), 0 < t < 1$

$\Rightarrow T = \frac{Y}{\theta} \sim Beta(1,2)$

$\because$ the pdf of Beta(1,2) is irrelevant to θ, $\therefore T = \frac{Y}{\theta}$ is a pivotal quantity.

1.(b)

$F_T(t) = \displaystyle\int_0^t 2(1-x)dx = 2x - x^2|_0^t = 2t - t^2$, $0 < t < 1$

1.(c)

Let a and b statisfy $F_T(a) = 0.05$ and $F_T(b) = 0.95$,

Then $0.9 = P(a < T < b) = P(a < \frac{Y}{\theta} < b) = P(\frac{Y}{b} < \theta < \frac{Y}{a})$

By (b), we can get a=0.025 and b=0.776, $[\dfrac{Y}{0.776}, \dfrac{Y}{0.025}]$ is a 90% Confidence Interval(CI) of θ.

2.(a)

We know $X \sim \chi^2(4)$, $f(x) = \frac{x}{4}e^{-\frac{x}{2}}$ and $Y \sim \Gamma(2, \lambda)$

Def. $Z = 2\lambda Y$, $y = \frac{z}{2\lambda}$, $J = \frac{1}{2\lambda}$

$f(y) = y\lambda^2 e^{-\lambda y}, y > 0$

$f(z) = (\dfrac{z}{2\lambda})\lambda^2 e^{-\lambda(\frac{z}{2\lambda})}(\dfrac{1}{2\lambda}) = \dfrac{z}{4}e^{-\frac{z}{2}}$

$\therefore X$ and Z have the same distribution.

2.(b)

$1 - \alpha = P(\chi^2_{\frac{\alpha}{2}}(4) \leqslant 2\lambda Y \leqslant \chi^2_{1-\frac{\alpha}{2}}(4))$

$P(\dfrac{\chi^2_{\frac{\alpha}{2}}(4)}{2Y} \leqslant \lambda \leqslant \dfrac{\chi^2_{1-\frac{\alpha}{2}}(4)}{2Y})$

Then the 95% C.I. for λ is $[\dfrac{\chi^2_{0.025}(4)}{2Y}, \dfrac{\chi^2_{0.975}(4)}{2Y}]$

3.(a)

$\because X \sim Beta(\alpha, \beta = 1)$ and $Y = -2\alpha \log X, |\dfrac{dx}{dy}| = \dfrac{1}{2\alpha}e^{\frac{-y}{2\alpha}}$

$\Rightarrow f(y) = \alpha e^{-\frac{y}{2\alpha}(\alpha-1)}\dfrac{1}{2\alpha}e^{-\frac{y}{2\alpha}} = \dfrac{1}{2}e^{\frac{-y}{2}} \therefore Y \sim Exp(\lambda = \dfrac{1}{2})$

3.(b)

$\because Y_i \sim Exp(\dfrac{1}{2}) \equiv \Gamma(1, \dfrac{1}{2})$, and $Y_1, ..., Y_n$ are independent

$\therefore \displaystyle\sum_{i=1}^{n} Y_i = -2\alpha \sum_{i=1}^{n} \log X_i \sim \Gamma(n, \dfrac{1}{2}) \equiv \chi^2(2n)$

$\because \sum_{i=1}^{n} Y_i$ a function of data $X_1, ..., X_n$ and parameter α, and its pdf is irrelevant to α

$\therefore \sum_{i=1}^{n} Y_i$ is a pivotal quantity.

3.(c)

$0.95 = P(\chi^2_{0.025}(2n) < -2\alpha \sum_{i=1}^{n} \log X_i < \chi^2_{0.975}(2n))$

$= P(\frac{\chi^2_{0.025}(2n)}{-2\sum_{i=1}^{n} \log X_i} < -2\alpha \sum_{i=1}^{n} \log X_i < \frac{\chi^2_{0.975}(2n)}{-2\sum_{i=1}^{n} \log X_i})$

$\therefore$ The 95% C.I. for α is $[\frac{\chi^2_{0.025}(2n)}{-2\sum_{i=1}^{n} \log X_i}, \frac{\chi^2_{0.975}(2n)}{-2\sum_{i=1}^{n} \log X_i}]$

4.

(f).

By L.N. CH8 p.83 $\sqrt{nI(\hat{\tau})}(\hat{\tau} - \tau) \xrightarrow{d} N(0,1)$

$\because \hat{\tau} = \overline{X}$ and $nI(\hat{\tau}) = E(-\frac{\partial^2 l(\underset{\sim}{X}|\tau)}{\partial \tau^2}) = \frac{n}{\hat{\tau}^2}$

$0.95 \approx P(Z_{0.025} < \sqrt{nI(\hat{\tau})}(\hat{\tau} - \tau) < Z_{0.975})$

$= P(Z_{0.025} < \frac{\sqrt{n}(\overline{X} - \tau)}{\overline{X}} < Z_{0.975})$

$= P(\overline{X} - \frac{Z_{0.975}\overline{X}}{\sqrt{n}} < \tau < \overline{X} - \frac{Z_{0.025}\overline{X}}{\sqrt{n}})$

$\therefore$ The 95% C.I. for τ is $[\overline{X} - \frac{Z_{0.975}\overline{X}}{\sqrt{n}}, \overline{X} - \frac{Z_{0.025}\overline{X}}{\sqrt{n}}]$

(g).

$\because \overline{X} \sim \frac{1}{n}\Gamma(n, \frac{1}{\tau}) => \frac{2\sum_{i=1}^{n} X_i}{\tau} \sim \Gamma(\frac{2n}{2}, \frac{1}{2}) \equiv \chi^2(2n)$

$\therefore P(\chi^2_{0.025}(2n) < \frac{2\sum_{i=1}^{n} X_i}{\tau} < \chi^2_{0.975}(2n)) = P(\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{0.975}(2n)} < \tau < \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{0.025}(2n)})$

$\therefore$ The 95% C.I. for τ is $[\frac{2\sum_{i=1}^{n} X_i}{\chi^2_{0.975}(2n)}, \frac{2\sum_{i=1}^{n} X_i}{\chi^2_{0.025}(2n)}]$