Chapter 9  Deflections of Beams

9.1 Introduction

in this chapter, we describe methods for determining the equation of the deflection curve of beams and finding deflection and slope at specific points along the axis of the beam

9.2 Differential Equations of the Deflection Curve

consider a cantilever beam with a concentrated load acting upward at the free end

the deflection $v$ is the displacement in the $y$ direction

the angle of rotation $\theta$ of the axis (also called slope) is the angle between the $x$ axis and the tangent to the deflection curve

point $m_1$ is located at distance $x$

point $m_2$ is located at distance $x + dx$

slope at $m_1$ is $\theta$

slope at $m_2$ is $\theta + d\theta$

denote $O'$ the center of curvature and $\rho$ the radius of curvature, then

$\rho \, d\theta = ds$

and the curvature $\kappa$ is
\[ \kappa = \frac{1}{\rho} = \frac{d\theta}{ds} \]

the sign convention is pictured in figure

slope of the deflection curve

\[ \frac{dv}{dx} = \tan \theta \quad \text{or} \quad \theta = \tan^{-1} \frac{dv}{dx} \]

for \( \theta \) small \( ds \approx dx \) \( \cos \theta \approx 1 \) \( \tan \theta \approx \theta \), then

\[ \kappa = \frac{1}{\rho} = \frac{d\theta}{dx} \quad \text{and} \quad \theta = \frac{dv}{dx} \]

\[ \kappa = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2v}{dx^2} \]

if the materials of the beam is linear elastic

\[ \kappa = \frac{1}{\rho} = \frac{M}{EI} \quad \text{[chapter 5]} \]

then the differential equation of the deflection curve is obtained

\[ \frac{d\theta}{dx} = \frac{d^2v}{dx^2} = \frac{M}{EI} \]

it can be integrated to find \( \theta \) and \( v \)

\[ \therefore \quad \frac{dM}{dx} = V \quad \frac{dV}{dx} = -q \]

then

\[ \frac{d^3v}{dx^3} = \frac{V}{EI} \quad \frac{d^4v}{dx^4} = -\frac{q}{EI} \]
sign conventions for $M$, $V$ and $q$ are shown
the above equations can be written in a simple form

$$EIt'' = M \quad EIt''' = V \quad EIt'''' = -q$$

this equations are valid only when Hooke's law applies and when the slope and the deflection are very small
for nonprismatic beam $[I = I(x)]$, the equations are

$$EI_x \frac{d^2v}{dx^2} = M$$

$$\frac{d}{dx} (EI_x \frac{d^2v}{dx^2}) = \frac{dM}{dx} = V$$

$$\frac{d^2}{dx^2} (EI_x \frac{d^2v}{dx^2}) = \frac{dV}{dx} = -q$$

the exact expression for curvature can be derived

$$\kappa = \frac{1}{\rho} = \frac{v''}{[1 + (v')^2]^{3/2}}$$

**9.3 Deflections by Integration of the Bending-Moment Equation**

substitute the expression of $M(x)$ into the deflection equation then integrating to satisfy

(i) boundary conditions
(ii) continuity conditions
(iii) symmetry conditions
to obtain the slope $\theta$ and the
deflection $v$ of the beam

this method is called method of successive integration

Example 9-1

determine the deflection of beam $AB$
supporting a uniform load of intensity $q$
also determine $\delta_{\text{max}}$ and $\theta_A, \theta_B$
flexural rigidity of the beam is $EI$
bending moment in the beam is

$$M = \frac{qLx}{2} - \frac{q x^2}{2}$$

differential equation of the deflection curve

$$EI v'' = \frac{qLx}{2} - \frac{q x^2}{2}$$

Then

$$EI v' = \frac{qLx^2}{4} - \frac{q x^3}{6} + C_1$$

$\therefore$ the beam is symmetry, $\therefore \theta = v' = 0$ at $x = L/2$

$$0 = \frac{qL(L/2)^2}{4} - \frac{q (L/2)^3}{6} + C_1$$
then \( C_1 = q L^3 / 24 \)

the equation of slope is

\[
v' = -\frac{q}{24 EI} (L^3 - 6L x^2 + 4x^3)
\]

integrating again, it is obtained

\[
v = -\frac{q}{24 EI} (L^3 x - 2L x^3 + x^4) + C_2
\]

boundary condition: \( v = 0 \) at \( x = 0 \)

thus we have \( C_2 = 0 \)

then the equation of deflection is

\[
v = -\frac{q}{24 EI} (L^3 x - 2L x^3 + x^4)
\]

maximum deflection \( \delta_{\text{max}} \) occurs at center \( x = L/2 \)

\[
\delta_{\text{max}} = -v\left(\frac{L}{2}\right) = -\frac{5 q L^4}{384 EI} \quad (\downarrow)
\]

the maximum angle of rotation occurs at the supports of the beam

\[
\theta_A = v'(0) = -\frac{q L^3}{24 EI} \quad (\mathfrak{A})
\]

and \( \theta_B = v'(L) = \frac{q L^3}{24 EI} \quad (\mathfrak{E}) \)
Example 9-2

determine the equation of deflection curve for a cantilever beam \( AB \) subjected to a uniform load of intensity \( q \)

also determine \( \theta_B \) and \( \delta_B \) at the free end

flexural rigidity of the beam is \( EI \)

bending moment in the beam

\[
M = -\frac{q L^2}{2} + q L x - \frac{q x^2}{2}
\]

\[
EI v'' = -\frac{q L^2}{2} + q L x - \frac{q x^2}{2}
\]

\[
EI v' = -\frac{q L^2 x}{2} + \frac{q L x^2}{2} - \frac{q x^3}{6} + C_1
\]

boundary condition \( v' = 0 = 0 \) at \( x = 0 \)

\[ C_1 = 0 \]

\[ v' = -\frac{q x}{6EI} (3 L^2 - 3 L x + x^2) \]

integrating again to obtain the deflection curve

\[ v = -\frac{q x^2}{24EI} (6 L^2 - 4 L x + x^2) + C_2 \]

boundary condition \( v = 0 \) at \( x = 0 \)

\[ C_2 = 0 \]
then

\[ v = - \frac{q x^2}{24EI} \left( 6 L^2 - 4 L x + x^2 \right) \]

\[ \theta_{\max} = \theta_B = v'(L) = - \frac{q L^3}{6 EI} \] (3)

\[ \delta_{\max} = - \delta_B = - v(L) = \frac{q L^4}{8 EI} \] (4)

Example 9-4

determine the equation of deflection curve, \( \theta_A, \theta_B, \delta_{\max} \) and \( \delta_C \)
flexural rigidity of the beam is \( EI \)
bending moments of the beam

\[ M = \frac{Pbx}{L} \quad (0 \leq x \leq a) \]

\[ M = \frac{Pbx}{L} - P (x - a) \quad (a \leq x \leq L) \]
differential equations of the deflection curve

\[ EI v'' = \frac{Pbx}{L} \quad (0 \leq x \leq a) \]

\[ EI v'' = \frac{Pbx}{L} - P (x - a) \quad (a \leq x \leq L) \]

integrating to obtain
\[
\begin{align*}
EIv' &= \frac{Pbx^2}{2L} + C_1 \quad (0 \leq x \leq a) \\
EIv' &= \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2 \quad (a \leq x \leq L)
\end{align*}
\]

2nd integration to obtain

\[
\begin{align*}
EIv &= \frac{Pbx^3}{6L} + C_1 x + C_3 \quad (0 \leq x \leq a) \\
EIv &= \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4 \quad (a \leq x \leq L)
\end{align*}
\]

boundary conditions

(i) \( v(0) = 0 \)  \hspace{1cm} (ii) \( y(L) = 0 \)

continuity conditions

(iii) \( v'(a^-) = v'(a^+) \)  \hspace{1cm} (iv) \( v(a^-) = v(a^+) \)

(i) \( v(0) = 0 \)  \Rightarrow  \( C_3 = 0 \)

(ii) \( v(L) = 0 \)  \Rightarrow  \( \frac{PbL^3}{6} - \frac{Pb^3}{6} + C_2 L + C_4 = 0 \)

(iii) \( v'(a^-) = v'(a^+) \)  \Rightarrow  \( \frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_2 \)

\( C_1 = C_2 \)

(iv) \( v(a^-) = v(a^+) \)  \Rightarrow  \( \frac{Pba^3}{6L} + C_1 a + C_3 = \frac{Pba^3}{6L} + C_2 a + C_4 \)

\( C_3 = C_4 \)
then we have

\[ C_1 = C_2 = -\frac{Pb (L^2 - b^2)}{6L} \]

\[ C_3 = C_4 = 0 \]

thus the equations of slope and deflection are

\[ v' = -\frac{Pb}{6LEI} (L^2 - b^2 - 3x^2) \quad (0 \leq x \leq a) \]

\[ v' = -\frac{Pb}{6LEI} (L^2 - b^2 - 3x^2) - \frac{P(x - a)^2}{2EI} \quad (a \leq x \leq L) \]

\[ v = -\frac{Pbx}{6LEI} (L^2 - b^2 - x^2) \quad (0 \leq x \leq a) \]

\[ v = -\frac{Pbx}{6LEI} (L^2 - b^2 - x^2) - \frac{P(x - a)^3}{6EI} \quad (a \leq x \leq L) \]

angles of rotation at supports

\[ \theta_A = v'(0) = -\frac{Pab(L + b)}{6LEI} \quad (\nabla) \]

\[ \theta_B = v'(L) = \frac{Pab(L + a)}{6LEI} \quad (\Box) \]

\[ \therefore \theta_A \text{ is function of } a \text{ (or } b), \text{ to find } (\theta_A)_{\text{max}}, \text{ set } d\theta_A / db = 0 \]

\[ \theta_A = -\frac{Pb(L^2 - b^2)}{6LEI} \]

\[ d\theta_A / db = 0 \quad \Rightarrow \quad L^2 - 3b^2 = 0 \quad \Rightarrow \quad b = \frac{L}{\sqrt{3}} \]
\[(\theta_A)_{\text{max}} = -\frac{PL^2\sqrt{3}}{27EI}\]

for maximum \(\delta\) occurs at \(x_1\), if \(a > b\), \(x_1 < a\)

\[
\frac{dv}{dx} = 0 \Rightarrow x_1 = \frac{L^2 - b^2}{3} \quad (a \geq b)
\]

\[
\delta_{\text{max}} = -\nu(x_1) = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EI} \quad (\downarrow)
\]

at \(x = L/2\) \(\delta_C = -\nu(L/2) = \frac{Pb(3L^2 - 4b^2)}{48EI} \quad (\downarrow)
\]

\(\therefore\) the maximum deflection always occurs near the midpoint, \(\therefore\) \(\delta_C\) gives a good approximation of the \(\delta_{\text{max}}\)

in most case, the error is less than 3%

an important special case is \(a = b = L/2\)

\[
\nu' = \frac{P}{16EI} (L^2 - 4x^2) \quad (0 \leq x \leq L/2)
\]

\[
\nu = \frac{P}{48EI} (3L^2 - 4x^2) \quad (0 \leq x \leq L/2)
\]

\(\nu'\) and \(\nu\) are symmetric with respect to \(x = L/2\)

\[
\theta_A = \theta_B = \frac{PL^3}{16EI}
\]

\[
\delta_{\text{max}} = \delta_C = \frac{PL^3}{48EI}
\]
9.4 Deflections by Integration of Shear-Force and Load Equations

The procedure is similar to that for the bending moment equation except that more integrations are required.

If we begin from the load equation, which is of fourth order, four integrations are needed.

Example 9-4

determine the equation of deflection curve for the cantilever beam $AB$ supporting a triangularly distributed load of maximum intensity $q_0$.

Also determine $\delta_B$ and $\theta_B$.

Flexural rigidity of the beam is $EI$.

$$q = \frac{q_0 (L - x)}{L}$$

$$EIv''' = -q = -\frac{q_0 (L - x)}{L}$$

The first integration gives

$$EIv'' = -\frac{q_0 (L - x)^2}{2L} + C_1$$

\[ \therefore v''(L) = V = 0 \Rightarrow C_1 = 0 \]

Thus

$$EIv'' = -\frac{q_0 (L - x)^2}{2L}$$
2nd integration

\[ EIv'' = -\frac{q_0 (L - x)^3}{6L} + C_2 \]

\[ \therefore v''(L) = M = 0 \Rightarrow C_2 = 0 \]

thus \[ EIv'' = -\frac{q_0 (L - x)^3}{6L} \]

3rd and 4th integrations to obtain the slope and deflection

\[ EIv' = -\frac{q_0 (L - x)^4}{24L} + C_3 \]

\[ EIv = -\frac{q_0 (L - x)^5}{120L} + C_3 x + C_4 \]

boundary conditions : \( v'(0) = v(0) = 0 \)

the constants \( C_3 \) and \( C_4 \) can be obtained

\[ C_3 = -\frac{q_0 L^3}{24} \quad C_4 = \frac{q_0 L^4}{120} \]

then the slope and deflection of the beam are

\[ v' = -\frac{q_0 x}{24LEI} (4L^3 - 6L^2 x + 4Lx^2 - x^3) \]

\[ v = -\frac{q_0 x^2}{120LEI} (10L^3 - 10L^2 x + 5Lx^2 - x^3) \]

\[ \theta_B = v'(L) = -\frac{q_0 L^3}{24 EI} \quad (\mathfrak{E}) \]
\[ \delta_B = -v(L) = \frac{q_0L^4}{30 \ EI} \quad (\downarrow) \]

Example 9-5

an overhanging beam \( ABC \) with a concentrated load \( P \) applied at the end
determine the equation of deflection curve and the deflection \( \delta_C \) at the end
flexural rigidity of the beam is \( EI \)

the shear forces in parts \( AB \) and \( BC \) are
\[ V = -\frac{P}{2} \quad (0 < x < L) \]
\[ V = P \quad (L < x < \frac{3L}{2}) \]

the third order differential equations are
\[ EIv''' = -\frac{P}{2} \quad (0 < x < L) \]
\[ EIv''' = P \quad (L < x < \frac{3L}{2}) \]

bending moment in the beam can be obtained by integration
\[ M = Ev'' = -\frac{Px}{2} + C_1 \quad (0 \leq x \leq L) \]
\[ M = Ev'' = Px + C_2 \quad (L \leq x \leq \frac{3L}{2}) \]
boundary conditions: \( v''(0) = v''(3L/2) = 0 \)

we get
\[
C_1 = 0 \quad C_2 = -\frac{3PL}{2}
\]

therefore the bending moment equations are
\[
M = Elv'' = -\frac{Px}{2} \quad (0 \leq x \leq L)
\]
\[
M = Elv'' = -\frac{P(3L - 2x)}{2} \quad (L \leq x \leq \frac{3L}{2})
\]

2\text{nd} integration to obtain the slope of the beam
\[
Elv' = -\frac{Px^2}{4} + C_3 \quad (0 \leq x \leq L)
\]
\[
Elv' = -\frac{Px(3L - x)}{2} + C_4 \quad (L \leq x \leq \frac{3L}{2})
\]

continuity condition: \( v'(L^-) = v'(L^+) \)
\[
-\frac{PL^2}{4} + C_3 = -PL^2 + C_4
\]

then \( C_4 = C_3 + \frac{3PL^2}{4} \)

the 3\text{rd} integration gives
\[
Elv = -\frac{Px^3}{12} + C_3x + C_5 \quad (0 \leq x \leq L)
\]
\[
Elv = -\frac{Px^2(9L - 2x)}{12} + C_4x + C_6 \quad (L \leq x \leq \frac{3L}{2})
\]
boundary conditions: \( v(0) = v(L) = 0 \)

we obtain

\[ C_5 = 0 \quad C_3 = \frac{PL^2}{12} \]

and then \( C_4 = \frac{5PL^2}{6} \)

the last boundary condition: \( v(L^+) = 0 \)

then \( C_6 = -\frac{PL^3}{4} \)

the deflection equations are obtained

\[
v = \frac{Px}{12EI} (L^2 - x^2) \quad (0 \leq x \leq L)
\]

\[
v = -\frac{P}{12EI} (3L^3 - 10L^2x + 9Lx^2 - 2x^3) \quad (L \leq x \leq \frac{3L}{2})
\]

\[
= -\frac{P}{12EI} (3L - x)(L - x)(L - 2x)
\]

deflection at \( C \) is

\[
\delta_C = -v\left(\frac{3L}{2}\right) = \frac{PL^3}{8EI} \quad (\downarrow)
\]

### 9.5 Method of Superposition

the slope and deflection of beam caused by several different loads acting simultaneously can be found by superimposing the slopes and deflections caused by the loads acting separately
consider a simply beam supports two loads: (1) uniform load of intensity \( q \) and (2) a concentrated load \( P \)

the slope and deflection due to load 1 are

\[
(\delta_C)_1 = \frac{5qL^4}{384EI}
\]

\[
(\theta_A)_1 = (\theta_B)_1 = \frac{qL^3}{24EI}
\]

the slope and deflection due to load 2 are

\[
(\delta_C)_2 = \frac{PL^3}{48EI}
\]

\[
(\theta_A)_2 = (\theta_B)_2 = \frac{PL^2}{16EI}
\]

therefore the deflection and slope due to the combined loading are

\[
\delta_C = (\delta_C)_1 + (\delta_C)_2 = \frac{5qL^4}{384EI} + \frac{PL^3}{48EI}
\]

\[
\theta_A = \theta_B = (\theta_A)_1 + (\delta_A)_2 = \frac{qL^3}{24EI} + \frac{PL^2}{16EI}
\]

tables of beam deflection for simply and cantilever beams are given in Appendix G

superposition method may also be used for distributed loading

consider a simple beam \( ACB \) with a triangular load acting on the left-hand half
the deflection of midpoint due to a concentrated load is obtained [table G-2]

\[ \delta_C = \frac{Pa}{48EI} (3L^2 - 4a^2) \]

substitute \( q \, dx \) for \( P \) and \( x \) for \( a \)

\[ d\delta_C = \frac{(qdx) \, x}{48EI} (3L^2 - 4x^2) \]

the intensity of the distributed load is

\[ q = \frac{2q_0x}{L} \]

then \( \delta_C \) due to the concentrated load \( q \)

\[ d\delta_C = \frac{q_0 \, x^2}{24LEI} (3L^2 - 4x^2) \, dx \]

thus \( \delta_C \) due to the entire triangular load is

\[ \delta_C = \frac{L^2}{6} \int_0^L \frac{q_0 \, x^2}{24LEI} (3L^2 - 4x^2) \, dx = \frac{q_0L^4}{240EI} \]

similarly, the slope \( \theta_A \) due to \( P \) acting on a distance \( a \) from left end is

\[ d\theta_A = \frac{Pab(L + b)}{6LEI} \]

replacing \( P \) with \( 2q_0x \, dx/L \), \( a \) with \( x \), and \( b \) with \( (L - x) \)
\[ d\theta_A = \frac{2q_0x^2(L - x)(L + L - x)}{6L^2EI} \, dx = \frac{q_0}{3L^2EI} (L - x)(2L - x)x^2 \, dx \]

thus the slope at \( A \) throughout the region of the load is

\[ \theta_A = \int_0^{L/2} \frac{q_0}{3L^2EI} (L - x)(2L - x)x^2 \, dx = \frac{41q_0L^3}{2880EI} \]

the principle of superposition is valid under the following conditions

1. Hooke's law holds for the material
2. the deflections and rotations are small
3. the presence of the deflection does not alter the actions of applied loads

these requirements ensure that the differential equations of the deflection curve are linear

Example 9-6

a cantilever beam \( AB \) supports a uniform load \( q \) and a concentrated load \( P \) as shown

determine \( \delta_B \) and \( \theta_B \)

\( EI = \) constant

from Appendix G :

for uniform load \( q \)

\[ (\delta_B)_1 = \frac{qa^3}{24EI} (4L - a) \quad (\theta_B)_1 = \frac{qa^3}{6EI} \]

for the concentrated load \( P \)
\[
(\delta_B)_2 = \frac{PL^3}{3EI}, \quad (\theta_B)_2 = \frac{PL^2}{2EI}
\]

then the deflection and slope due to combined loading are

\[
\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qa^3}{24EI} (4L - a) + \frac{PL^3}{3EI}
\]

\[
\theta_B = (\theta_B)_1 + (\theta_B)_2 = \frac{qa^3}{6EI} + \frac{PL^2}{2EI}
\]

Example 9-7

a cantilever beam \( AB \) with uniform load \( q \) acting on the right-half
determine \( \delta_B \) and \( \theta_B \)
\( EI = \text{constant} \)
consider an element of load has magnitude \( q \, dx \) and is located at distance \( x \) from the support
from Appendix G, table G-1, case 5
by replacing \( P \) with \( q \, dx \) and \( a \) with \( x \)
\[
d\delta_B = \frac{(qdx)(x^2)(3L-x)}{6EI} \quad d\theta_B = \frac{(qdx)(x^2)}{2EI}
\]
by integrating over the loaded region
\[
\delta_B = \int_{L/2}^{L} \frac{qx^2(3L-x)}{6EI} \, dx = \frac{41qL^4}{384EI}
\]
\[ \theta_B = \frac{L^2}{2L/2} \frac{qx^2}{2EI} dx = \frac{7qL^3}{48EI} \]

Example 9-8

A compound beam \( ABC \) supports a concentrated load \( P \) and an uniform load \( q \) as shown.

determine \( \delta_B \) and \( \theta_A \)

\( EI = \) constant

we may consider the beam as composed of two parts: (1) simple beam \( AB \), and (2) cantilever beam \( BC \)

the internal force \( F = 2P/3 \) is obtained for the cantilever beam \( BC \)

\[ \delta_B = \frac{qb^4}{8EI} + \frac{Fb^3}{3EI} = \frac{qb^4}{8EI} + \frac{2Pb^3}{9EI} \]

for the beam \( AB \), \( \theta_A \) consists of two parts: (1) angle \( BAB' \) produced by \( \delta_B \), and (2) the bending of beam \( AB \) by the load \( P \)

\[ (\theta_A)_1 = \frac{\delta_B}{a} = \frac{qb^4}{8aEI} + \frac{2Pb^3}{9aEI} \]

the angle due to \( P \) is obtained from Case 5 of table G-2, Appendix G with replacing \( a \) by \( 2a/3 \) and \( b \) by \( a/3 \)
\[
(\theta_A)_2 = \frac{P(2a/3)(a/3)(a + a/3)}{6aEI} = \frac{4Pa^2}{81EI}
\]

note that in this problem, \(\delta_B\) is continuous and \(\theta_B\) does not continuous, i.e. \((\theta_B)_L \neq (\theta_B)_R\)

Example 9-9

an overhanging beam \(ABC\) supports a uniform load \(q\) as shown
determine \(\delta_C\)
\(EI\) = constant

\(\delta_C\) may be obtained due to two parts
(1) rotation of angle \(\theta_B\)
(2) a cantilever beam subjected
to uniform load \(q\)

firstly, we want to find \(\theta_B\)

\[
\theta_B = \frac{-qL^3}{24EI} + \frac{M_BL}{3EI}
\]

\[
= \frac{-qL^3}{24EI} + \frac{qa^2L}{6EI} = \frac{qL(4a^2 - L^2)}{24EI}
\]

then \(\delta_1 = a\theta_B = \frac{qaL(4a^2 - L^2)}{24EI}\)

bending of the overhang \(BC\) produces an additional deflection \(\delta_2\)

\[
\delta_2 = \frac{qa^4}{8EI}
\]
therefore, the total downward deflection of \( C \) is

\[
\delta_C = \delta_1 + \delta_1 = \frac{qaL(4a^2 - L^2)}{24EI} + \frac{qa^4}{8EI}
\]

\[
= \frac{qa}{24EI} (a + L)(3a^2 + aL - L^2)
\]

for \( a \) large, \( \delta_C \) is downward; for \( a \) small, \( \delta_C \) is upward

for \( \delta_C = 0 \) \( 3a^2 + aL - L^2 = 0 \)

\[
a = \frac{L(\sqrt{13} - 1)}{6} = 0.4343L
\]

\( a > 0.4343L, \ \delta_C \) is downward; \( a < 0.4343L, \ \delta_C \) is upward

point \( D \) is the point of inflection, the curvature is zero because the bending moment is zero at this point

at point \( D, \ \frac{d^2y}{dx^2} = 0 \)

**9.6 Moment-Area Method**

the method is especially suitable when the deflection or angle of rotation at only one point of the beam is desired

consider a segment \( AB \) of the beam
denote \( \theta_{B/A} \) the difference between \( \theta_B \) and \( \theta_A \)

\[
\theta_{B/A} = \theta_B - \theta_A
\]

consider points \( m_1 \) and \( m_2 \)
\[ d\theta = \frac{ds}{\rho} = \frac{Mdx}{EI} \]

\( Mdx / EI \) is the area of the shaded strip of the \( Mdx / EI \) diagram integrating both side between \( A \) and \( B \)

\[
\int_{A}^{B} d\theta = \int_{A}^{B} \frac{M}{EI} \, dx
\]

\( \theta_{B/A} = \theta_{B} - \theta_{A} = \) area of the \( M/EI \) diagram between \( A \) and \( B \)

this is the \textbf{First moment-area theorem}

next, let us consider the vertical offset \( t_{B/A} \) between points \( B \) and \( B_{1} \) (on the tangent of \( A \))

\[ \therefore \ dt = x_{1} \, d\theta = x_{1} \frac{Mdx}{EI} \]

integrating between \( A \) and \( B \)

\[
\int_{A}^{B} dt = \int_{A}^{B} x_{1} \frac{Mdx}{EI}
\]

i.e. \( t_{B/A} = 1^{\text{st}} \) moment of the area of the \( M/EI \) diagram between \( A \) and \( B \), evaluated w. r. t. \( B \)

this if the \textbf{Second moment-area theorem}

Example 9-10

determine \( \theta_{B} \) and \( \delta_{B} \) of a cantilever beam \( AB \) supporting a
concentrated load $P$ at $B$

sketch the $M/EI$ diagram first

$$A_1 = -\frac{1}{2} L \frac{PL}{EI} = -\frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = \theta_B = -\frac{PL^2}{2EI}$$

$$Q_1 = A_1 x = A_1 \frac{2L}{3} = -\frac{PL^3}{6EI}$$

$$\delta_B = -Q_1 = -\frac{PL^3}{6EI}$$

Example 9-11

determine $\theta_B$ and $\delta_B$ of a cantilever beam $AB$ supporting a uniform load of intensity $q$ acting over the right-half

sketch the $M/EI$ diagram first

$$A_1 = \frac{1}{3} L \left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{48EI}$$

$$A_2 = \frac{L}{2} \left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{16EI}$$

$$A_3 = \frac{1}{2} L \left(\frac{qL^2}{4EI}\right) = \frac{qL^3}{16EI}$$
\[ \theta_{B/A} = \theta_B = A_1 + A_2 + A_3 = \frac{7qL^3}{16EI} \]  
\[ \delta_B = t_{B/A} = A_1 x_1 + A_2 x_2 + A_3 x_3 = \frac{qL^3}{EI} \left( \frac{1}{48} \frac{3L}{8} + \frac{1}{16} \frac{3L}{4} + \frac{1}{16} \frac{5L}{6} \right) = \frac{41qL^4}{384EI} \]

Example 9-12

A simple beam \( ADB \) supports a concentrated load \( P \) as shown. Determine \( \theta_A \) and \( \delta_D \)

\[ A_1 = \frac{L}{2} \left( \frac{P_{ab}}{LEI} \right) = \frac{P_{ab}}{2EI} \]

\[ t_{B/A} = A_1 \frac{L + b}{3} = \frac{P_{ab} (L + b)}{6EI} \]

\[ \theta_A = \frac{BB_1}{L} = \frac{P_{ab} (L + b)}{6EIL} \]

to find the deflection \( \delta_D \) at \( D \)

\[ \delta_D = DD_1 - D_2D_1 \]

\[ DD_1 = a \theta_A = \frac{Pa^2b (L + b)}{6EIL} \]

\[ D_2D_1 = t_{D/A} = A_2 x_2 = \frac{A P_{ab} a}{2 EIL \frac{3}{3}} = \frac{Pa^3b}{6EIL} \]
\[ \delta_D = \frac{Pa^2b^2}{3EIL} \]

to find the maximum deflection \( \delta_{\text{max}} \) at \( E \), we set \( \theta_E = 0 \)

\[ A_3 = \frac{x_1 Pbx_1}{2 EIL} = \frac{Pbx_1^2}{2EIL} \]

\[ \theta_{E/A} = \theta_E - \theta_A = -A_3 = - \frac{Pbx_1^2}{2EIL} \]

\[ \theta_A = \frac{Pab (L + b)}{6EIL} = \frac{Pbx_1^2}{2EIL} \]

then \( x_1 = \left[ \frac{(L^2 - b^2)}{3} \right]^{1/2} \)

and \( \delta_{\text{max}} = x_1 \theta_A - A_3 \frac{x_1}{3} = \frac{Pb}{9 \sqrt{3} EIL} (L^2 - b^2)^{3/2} \)

or \( \delta_{\text{max}} = \) offset of point \( A \) from tangent at \( E \)

\[ \delta_{\text{max}} = A_3 \frac{2 x_1}{3} = \frac{Pb}{9 \sqrt{3} EIL} (L^2 - b^2)^{3/2} \]

Conjugate Beam Method

\[ EIv'' = EI \frac{d\theta}{dx} = M \]

Integrating

\[ \theta = \int \frac{M}{EI} dx \]

\[ v = \int \int \frac{M}{EI} dx \, dx \]
beam theory

\[ \frac{dM}{dx} = V \quad \frac{dV}{dx} = -q \]

\[ V = -\int q \, dx \quad M = -\int \int q \, dx \, dx \]

suppose we have a beam, called conjugate beam, whose length equal to the actual beam, let this beam subjected to so-called "elastic load" of intensity \( M/EI \), then the shear force and bending moment over a portion of the conjugate beam, denoted by \( V \) and \( M \), can be obtained

\[ V = -\int \frac{M}{EI} \, dx \quad M = -\int \int \frac{M}{EI} \, dx \, dx \]

then

(1) the slope at the given section of the beam equals the minus shear force in the corresponding section of the conjugate beam

(2) the deflection at the given section of the beam equals the minus bending moment in the corresponding section of the conjugate beam

i.e. \( \theta = -V \)

\( \delta = -M \)

the support conditions between the actual beam and conjugate beam can be found

<table>
<thead>
<tr>
<th>Actual Beam</th>
<th>Conjugate Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>fixed end</strong></td>
<td>( \theta = 0, \quad v = 0 )</td>
</tr>
<tr>
<td><strong>free end</strong></td>
<td>( \theta \neq 0, \quad v \neq 0 )</td>
</tr>
<tr>
<td><strong>simple end</strong></td>
<td>( \theta \neq 0, \quad v = 0 )</td>
</tr>
<tr>
<td><strong>interior support</strong></td>
<td>( \theta \neq 0, \quad v = 0 )</td>
</tr>
<tr>
<td><strong>interior hinge</strong></td>
<td>( \theta \neq 0, \quad v \neq 0 )</td>
</tr>
</tbody>
</table>
Example 1

\[
\theta_B = -V_B = \frac{1}{2} \frac{PL}{EI} L = -\frac{PL^2}{2EI} \quad (\uparrow)
\]

\[
\delta_B = -M_B = \frac{PL^2}{2EI} \frac{2L}{3} = -\frac{PL^3}{3EI} \quad (\downarrow)
\]

Example 2

\[
\theta_A = -V_A = \frac{1}{2} \frac{2L w L^2}{3 \ 8EI} = -\frac{w L^3}{24EI} \quad (\uparrow)
\]

\[
\delta_C = -M_C = -\frac{w L^3}{24EI} \frac{L}{2} - \frac{w L^2}{8EI} \frac{L}{2} \frac{L}{4}
+ \frac{1}{3} \frac{w L^2}{8EI} \frac{L}{2} \frac{3L}{8} = -\left( \frac{1}{48} + \frac{1}{64} \right) \frac{w L^4}{EI}
= \frac{5}{384EI} \quad (\downarrow)
\]

28
Example 3

\[ \theta_A = - \frac{V_A}{R_A} = - \frac{1}{2} \frac{M}{EI} \frac{2L}{3} = - \frac{ML}{3EI} \quad (\text{a}) \]

\[ \theta_B = - \frac{V_B}{R_B} = \frac{1}{2} \frac{M}{EI} \frac{L}{3} = \frac{ML}{6EI} \quad (\text{c}) \]

\[ \delta_C = - \frac{M_C}{2EI} = - \left( \frac{ML}{6EI} \frac{L}{2} \frac{M}{2EI} \frac{L}{6} \right) \]

\[ = - \frac{1}{12} \frac{ML^2}{EI} = \frac{ML^2}{16EI} \quad (\downarrow) \]

Example 4

\[ \theta_B = - \frac{V_B}{\bar{R}_B} = \frac{ML}{EI} \quad (\text{a}) \]

\[ \delta_B = - \frac{M_B}{EI} = \frac{ML}{2EI} \quad (\uparrow) \]

Example 5

\[ \theta_B = - \frac{V_B}{R_B} = - \frac{L}{3} \frac{qL^2}{2EI} = - \frac{qL^3}{6EI} \quad (\text{a}) \]

\[ \delta_B = - \frac{M_B}{EI} = - \frac{qL^3}{6EI} \frac{3L}{4} = - \frac{qL^4}{8EI} \quad (\downarrow) \]
Example 6

\[
\theta_A = -\frac{V_A}{2} = -\frac{1}{2} \frac{L \cdot PL}{4EI} = -\frac{PL^2}{16EI} \quad (3)
\]

\[
\delta_C = -\frac{M_C}{EI} = -\left(\frac{PL^2}{16EI} \frac{L}{2} \frac{1}{2} \frac{L}{2} \frac{PL}{6}\right)
\]

\[
= -\frac{PL^3}{EI} \left(\frac{1}{32} \frac{1}{96}\right) = -\frac{PL^3}{48EI} \quad (\downarrow)
\]

9.7 Nonprismatic Beam

\(EI \neq \text{constant}\)

Example 9-13

A beam \(ABCDE\) is supported a concentrated load \(P\) at midspan as shown

\[I_{BD} = 2I_{AB} = 2I_{DE} = 2I\]

determine the deflection curve, \(\theta_A\) and \(\delta_C\)

\[
M = \frac{Px}{2} \quad (0 \leq x \leq \frac{L}{2})
\]

then \(EIv'' = \frac{Px}{2} \quad (0 \leq x \leq L/4)\)

\(E(2I)v'' = \frac{Px}{2} \quad (L/4 \leq x \leq L/2)\)
thus \( v' = \frac{P x^2}{4EI} + C_1 \quad (0 \leq x \leq L/4) \)

\[ v' = \frac{P x^2}{8EI} + C_2 \quad (L/4 \leq x \leq L/2) \]

\[ \therefore \quad v' = 0 \quad \text{at} \quad x = L/2 \quad \text{(symmetric)} \]

\[ \therefore \quad C_2 = -\frac{PL^2}{32EI} \]

continuity condition \( v'(L/4)^- = v'(L/4)^+ \)

\[ C_1 = -\frac{5PL^2}{128EI} \]

therefore \( v' = -\frac{P}{128EI} (5L^2 - 32x^2) \quad (0 \leq x \leq L/4) \)

\[ v' = -\frac{P}{32EI} (L^2 - 4x^2) \quad (L/4 \leq x \leq L/2) \]

the angle of rotation \( \theta_A \) is

\[ \theta_A = v'(0) = -\frac{5PL^2}{128EI} \quad (\theta) \]

integrating the slope equation and obtained

\[ v = -\frac{P}{128EI} (5L^2 x - \frac{32x^3}{3}) + C_3 \quad (0 \leq x \leq L/4) \]

\[ v = -\frac{P}{32EI} (L^2 x - \frac{4x^3}{3}) + C_4 \quad (L/4 \leq x \leq L/2) \]
boundary condition \( v(0) = 0 \)

we get \( C_3 = 0 \)

continuity condition \( v(L/4)^- = v(L/4)^+ \)

we get \( C_4 = -\frac{PL^3}{768EI} \)

therefore the deflection equations are

\[
v = -\frac{Px}{384EI} (15L^2 - 32x^2) \quad (0 \leq x \leq L/4)
\]

\[
v = -\frac{P}{768EI} (L^3 + 24L^2x - 32x^3) \quad (L/4 \leq x \leq L/2)
\]

the midpoint deflection is obtained

\[
\delta_C = -v(L/2) = \frac{3PL^3}{256EI} \quad (\downarrow)
\]

moment-area method and conjugate beam methods can also be used

Example 9-14

a cantilever beam \( ABC \) supports a concentrated load \( P \) at the free end

\( I_{BC} = 2 I_{AB} = 2I \)

determine \( \delta_A \)

denote \( \delta_1 \) the deflection of \( A \) due to \( C \) fixed

\[
\delta_1 = \frac{P(L/2)^3}{3EI} = \frac{PL^3}{24EI}
\]
and \( \delta_c = \frac{P(L/2)^3}{3E(2I)} + \frac{(PL/2)(L/2)^2}{2E(2I)} = \frac{5PL^3}{96EI} \)

\( \theta_c = \frac{P(L/2)^2}{2E(2I)} + \frac{(PL/2)(L/2)}{E(2I)} = \frac{PL^2}{16EI} \)

addition deflection at \( A \) due to \( \delta_c \) and \( \theta_c \)

\[ \delta_2 = \delta_c + \frac{\theta_c L}{2} = \frac{5PL^3}{48EI} \]

\[ \delta_A = \delta_1 + \delta_2 = \frac{5PL^3}{16EI} \]

moment-area method and conjugate beam methods can also be used

9.8 Strain Energy of Bending

consider a simple beam \( AB \) subjected to pure bending under the action of two couples \( M \)
the angle \( \theta \) is

\[ \theta = \frac{L}{\rho} = \kappa L = \frac{ML}{EI} \]

if the material is linear elastic, \( M \) and \( \theta \) has linear relation, then

\[ W = U = \frac{M\theta}{2} = \frac{M^2L}{2EI} = \frac{EI\theta^2}{2L} \]
for an element of the beam

\[ d\theta = \kappa dx = \frac{d^2y}{dx^2} \]

\[ dU = dW = \frac{Md\theta}{2} = \frac{M^2dx}{2EI} = \frac{EI(d\theta)^2}{2dx} \]

by integrating throughout the length of the beam

\[ U = \int_0^L \frac{M^2dx}{2EI} = \int_0^L \frac{EI}{2} \left( \frac{d^2y}{dx^2} \right)^2 dx \]

shear force in beam may produce energy, but for the beam with \( L/d > 8 \), the strain energy due to shear is relatively small and may be disregarded

deflection caused by a single load

\[ U = W = \frac{P\delta}{2} \quad U = W = \frac{M_0\theta}{2} \]

\[ \delta = \frac{2U}{P} \quad \text{or} \quad \theta = \frac{2U}{M_0} \]

Example 9-15

a simple beam \( AB \) of length \( L \) supports a uniform load of intensity \( q \)
evaluate the strain energy

\[ M = \frac{qLx}{2} - \frac{qx^2}{2} = \frac{q}{2} (Lx - x^2) \]
\[ U = \int_0^L \frac{M^2}{2EI} \, dx = \frac{1}{2EI} \int_0^L \frac{q^2}{2} \left[(Lx - x^2)\right]^2 \, dx \]

\[ = \frac{q^2}{8EI} \int_0^L (L^2x^2 - 2Lx^3 + x^4) \, dx = \frac{q^2L^5}{240EI} \]

Example 9-16

a cantilever beam \( AB \) is subjected to three different loading conditions

(a) a concentrated load \( P \) at its free end

(b) a moment \( M_0 \) at its free end

(c) both \( P \) and \( M_0 \) acting simultaneously
determine \( \delta_A \) due to loading (a)
determine \( \theta_A \) due to loading (b)

(a) \( M = -Px \)

\[ U = \int_0^L \frac{M^2}{2EI} \, dx = \int_0^L \frac{(-Px)^2}{2EI} \, dx = \frac{F}{6} \]

\[ W = U = \frac{P\delta_A}{2} = \frac{P^2L^3}{6EI} \quad \delta_A = \frac{PL^3}{3EI} \]

(b) \( M = -M_0 \)

\[ U = \int_0^L \frac{M^2}{2EI} \, dx = \int_0^L \frac{(-M_0)^2}{2EI} \, dx = \frac{M_0^2L}{2EI} \]

\[ W = U = \frac{M_0\theta_A}{2} = \frac{M_0^2L}{2EI} \quad \theta_A = \frac{M_0L}{EI} \]
(c) \[ M = -P\delta - M_0 \]

\[ U = \int_0^L \frac{M^2}{2EI} \, dx = \int_0^L (-P\delta - M_0)^2 \, dx = \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI} \]

\[ W = U \ldots \]

1 equation for two unknowns \( \delta_A \) and \( \theta_A \)

9.9 Castigliano's Theorem (Energy Method)

\[ dU = P \, d\delta \quad \frac{dU}{d\delta} = P \]

\[ dC = \delta \, dP \quad \frac{dC}{dP} = \delta \]

where \( C \) is complementary strain energy

for linear elastic materials \( C = U \)

then we have \( \frac{dU}{dP} = \delta \)

similarly \( \frac{dU}{dM} = \theta \)

for both \( P \) and \( M \) acting simultaneously, \( U = U(P, M) \)

\[ \frac{\partial U}{\partial P} = \delta \quad \frac{\partial U}{\partial M} = 0 \]

in example 9-16 (c)
\[
U = \frac{P^2 L^3}{6EI} + \frac{PM_0 L^2}{2EI} + \frac{M_0^2 L}{2EI}
\]

\[
\delta = \frac{\partial U}{\partial P} = \frac{PL^3}{6EI} + \frac{M_0 L^2}{2EI}
\]

\[
\theta = \frac{\partial U}{\partial M} = \frac{PL^2}{2EI} + \frac{M_0 L}{EI}
\]

in general relationship

\[
\delta_i = \frac{\partial U}{\partial P_i} \quad \text{Castigliano's Theorem}
\]

\[
\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i} \int \frac{M^2 dx}{2EI} = \int \frac{M \frac{\partial M}{\partial P_i}}{EI} dx
\]

this is the modified Castigliano's Theorem

in example 9-16 (c)

\[
M = -Px - M_0
\]

\[
\frac{\partial M}{\partial P} = -x \quad \frac{\partial U}{\partial M_0} = -1
\]

\[
\delta = \frac{1}{EI} \int (-Px - M_0)(-x) dx = \frac{PL^3}{6EI} + \frac{M_0 L^2}{2EI}
\]

\[
\theta = \frac{1}{EI} \int (-Px - M_0)(-1) dx = \frac{PL^2}{2EI} + \frac{M_0 L}{EI}
\]
Example 9-17

a simple beam $AB$ supports a uniform load $q = 20$ kN/m, and a concentrated load $P = 25$ kN

$L = 2.5$ m $E = 210$ GPa

$I = 31.2 \times 10^2$ cm$^4$

determine $\delta_C$

$$M = \frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2}$$

method (1)

$$U = \int \frac{M^2 dx}{2EI} = 2 \int_0^{L/2} \frac{1}{2EI} (\frac{Px}{2} + \frac{qLx}{2} - \frac{qx^2}{2})^2 dx$$

$$= \frac{P^2L^3}{96EI} + \frac{5PqL^4}{384EI} + \frac{q^2L^5}{240EI}$$

$$\delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{5qL^4}{384EI}$$

method (2)

$$\frac{\partial M}{\partial P} = \frac{x}{2}$$

$$\delta_C = \int \frac{M}{EI} \frac{\partial M}{\partial P} dx = 2 \int_0^{L/2} \frac{Px}{2EI} (\frac{qLx}{2} - \frac{qx^2}{2})^2 dx$$

$$= \frac{PL^3}{48EI} + \frac{5qL^4}{384EI}$$

$$= 1.24$ mm $+ 1.55$ mm $= 2.79$ mm
Example 9-18

a overhanging beam $ABC$ supports a uniform load and a concentrated load as shown
determine $\delta_C$ and $\theta_C$

the reaction at $A$ due to the loading is

$$R_A = \frac{qL}{2} - \frac{P}{2}$$

$$M_{AB} = R_A x_1 - \frac{q x_1^2}{2}$$

$$M_{AB} = \frac{q L x_1}{2} - \frac{P x_1}{2} - \frac{q x_1^2}{2} \quad \text{(}0 \leq x_1 \leq L\text{)}$$

$$M_{BC} = -P x_2 \quad \text{(}0 \leq x_1 \leq L/2\text{)}$$

then the partial derivatives are

$$\frac{\partial M_{AB}}{\partial P} = -x_1/2 \quad \text{(}0 \leq x_1 \leq L\text{)}$$

$$\frac{\partial M_{BC}}{\partial P} = -x_2 \quad \text{(}0 \leq x_2 \leq L/2\text{)}$$

$$\delta_C = \int (M/EI) (\frac{\partial M}{\partial P}) dx$$

$$\delta_C = \int_0^L \left( \frac{M_{AB}}{EI} \right) (\frac{\partial M_{AB}}{\partial P}) dx + \int_0^{L/2} \left( \frac{M_{BC}}{EI} \right) (\frac{\partial M_{BC}}{\partial P}) dx$$

$$\delta_C = \frac{1}{EI} \int_0^L \left( \frac{q L x_1}{2} - \frac{P x_1}{2} - \frac{q x_1^2}{2} \right) (-x_1) dx_1 + \frac{1}{EI} \int_0^{L/2} (-P x_2)(-x_2) dx_2$$

$$\delta_C = \frac{PL^3}{8EI} - \frac{q L^4}{48EI}$$

39
to determine the angle $\theta_C$, we place a couple of moment $M_C$ at $C$

$$R_A = \frac{qL}{2} - \frac{P}{2} - \frac{M_C}{L}$$

$$M_{AB} = R_A x_1 - \frac{q x_1^2}{2}$$

$$= \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_Cx_1}{L} - \frac{q x_1^2}{2} \quad (0 \leq x_1 \leq L)$$

$$M_{BC} = -Px_2 - M_C \quad (0 \leq x_2 \leq L/2)$$

then the partial derivatives are

$$\frac{\partial M_{AB}}{\partial M_C} = -x_1/L \quad (0 \leq x_1 \leq L)$$

$$\frac{\partial M_{BC}}{\partial M_C} = -1 \quad (0 \leq x_2 \leq L/2)$$

$$\theta_C = \int (M/EI)(M/M_C)dx$$

$$= \int_0^L (M_{AB}/EI)(M_{AB}/M_C)dx + \int_0^{L/2} (M_{BC}/EI)(M_{BC}/M_C)dx$$

$$= \frac{1}{EI} \left[ \int_0^L \left( \frac{qLx_1}{2} - \frac{Px_1}{2} - \frac{M_Cx_1}{L} - \frac{q x_1^2}{2} \right) \left( -\frac{x_1}{L} \right) dx_1 \right]$$

$$+ \frac{1}{EI} \int_0^{L/2} (-Px_2 - M_C)(-1)dx_2$$

since $M_C$ is a virtual load, set $M_C = 0$, after integrating $\theta_C$ is obtained

$$\theta_C = \frac{7PL^2}{24EI} - \frac{qL^4}{24EI}$$
9.10 Deflections Produced by Impact

9.11 Temperature Effects