Chapter 7  Analysis of Stresses and Strains

7.1 Introduction

axial load  \( \sigma = \frac{P}{A} \)

torsional load in circular shaft  \( \tau = \frac{T \rho}{I_p} \)

bending moment and shear force in beam

\[
\sigma = \frac{M y}{I} \quad \tau = \frac{V Q}{I b}
\]

in this chapter, we want to find the normal and shear stresses acting on any inclined section

for uniaxial load and pure shear, this relation are shown in chapters 2 and 3, now we want to derive the transformation relationships that give the stress components for any orientation

this is referred as stress transformation

when an element is rotated from one orientation to another, the stresses acting on the faces of the element are different but they still represent the same state of stress, namely, the stress at the point under consideration

7.2 Plane Stress

consider the infinitesimal element with its edges parallel to \( x, y, \) and \( z \) axes

if only the \( x \) and \( y \) faces of the element are subjected to stresses, it is called plane stress, it can be shown as a two dimension stress element
equal normal stresses act on opposite faces, shear stress \( \tau \) has two subscripts, the first denotes the face on which the stress acts, and the second gives the direction of that face

\( \tau_{xy} : \) acts on \( x \) face directed to \( y \) axis  
\( \tau_{yx} : \) acts on \( y \) face directed to \( x \) axis  

sign convention: acts on \( \pm \) face and directed to \( \pm \) axis as \( + \)

as discussed in chapter 1, \( \tau_{xy} = \tau_{yx} \)

consider an element located at the same point and whose faces are perpendicular to \( x_1, y_1 \) and \( z_1 \) axes, in which \( z_1 \) axis coincides with the \( z \) axis, and \( x_1 \) and \( y_1 \) axes are rotated counterclockwise through an angle \( \theta \) w.r.t. \( x \) and \( y \) axes, the normal and shear stresses acting on this new element are denoted \( \sigma_{x_1}, \sigma_{y_1}, \tau_{x_1y_1}, \text{ and } \tau_{y_1x_1} \)

also \( \tau_{x_1y_1} = \tau_{y_1x_1} \)

the stresses acting on the rotated \( x_1y_1 \) element can be expressed in terms of stress on the \( xy \) element by using equation of static equilibrium

choose a wedge-shaped element

force equilibrium in \( x_1 \)-direction

\[
\begin{align*}
\sigma_{x_1} A_0 \sec \theta & - \sigma_x A_0 \cos \theta - \tau_{xy} A_0 \sin \theta \\
& - \sigma_y A_0 \tan \theta \sin \theta - \tau_{yx} A_0 \tan \theta \cos \theta = 0
\end{align*}
\]

force equilibrium in \( y_1 \)-direction

\[
\begin{align*}
\tau_{x_1y_1} A_0 \sec \theta & + \sigma_x A_0 \sin \theta - \tau_{xy} A_0 \cos \theta
\end{align*}
\]
\[- \sigma_y A_0 \tan \theta \cos \theta + \tau_{yx} A_0 \tan \theta \sin \theta = 0\]

with \(\tau_{xy} = \tau_{yx}\)

it is obtained

\[
\begin{align*}
\sigma_{x_1} &= \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \\
\tau_{x_1y_1} &= - (\sigma_x - \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)
\end{align*}
\]

for \(\theta = 0^\circ\) \(\sigma_{x_1} = \sigma_x\) \(\tau_{x_1y_1} = \tau_{xy}\)

for \(\theta = 90^\circ\) \(\sigma_{x_1} = \sigma_y\) \(\tau_{x_1y_1} = - \tau_{xy}\)

from trigonometric identities

\[
\begin{align*}
\cos^2 \theta &= \frac{1}{2} (1 + \cos 2\theta) \\
\sin^2 \theta &= \frac{1}{2} (1 - \cos 2\theta) \\
\sin \theta \cos \theta &= \frac{1}{2} \sin 2\theta
\end{align*}
\]

the above equations can be expressed in a more convenient form

\[
\begin{align*}
\sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\tau_{x_1y_1} &= - \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\end{align*}
\]

this is the **transformation equations for plane stress**

substituting \(\theta + 90^\circ\) for \(\theta\) in \(\sigma_{x_1}\) equation

\[
\begin{align*}
\sigma_{y_1} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta
\end{align*}
\]

also we obtain the following equation for plane stress
\[
\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y
\]
i.e. the sum of the normal stresses acting on perpendicular faces for a plane stress element is constant, independent of \( \theta \)
\( \sigma_{x1} \) and \( \tau_{x1y1} \) versus the angle of rotation \( \theta \) can be plotted as

for uniaxial stress case, \( \sigma_y = 0, \tau_{xy} = 0 \)
\[
\sigma_{x1} = \sigma_x \cos^2 \theta = \sigma_x (1 + \cos 2\theta) / 2
\]
\[
\tau_{x1y1} = -\sigma_x \sin \theta \cos \theta = -\sigma_x \sin 2\theta / 2
\]
for pure shear stress case, \( \sigma_x = \sigma_y = 0 \)
\[
\sigma_{x1} = 2 \tau_{xy} \sin \theta \cos \theta = \tau_{xy} \sin 2\theta
\]
\[
\tau_{x1y1} = \tau_{xy} (\cos^2 \theta - \sin^2 \theta) = \tau_{xy} \cos 2\theta
\]
same as derived in previous chapters
for biaxial stress case, \( \tau_{xy} = 0 \)
\[
\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta
\]
\[
\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta
\]

Example 7-1
\[
\sigma_x = 110 \text{ MPa} \quad \sigma_y = 40 \text{ MPa}
\]
\[
\tau_{xy} = \tau_{yx} = 28 \text{ MPa}
\]
determine the stresses for \( \theta = 45^\circ \)
\[
\frac{\sigma_x + \sigma_y}{2} = 75 \text{ MPa} \quad \frac{\sigma_x - \sigma_y}{2} = 35 \text{ MPa}
\]
\[
\sin 2\theta = \sin 90^\circ = 1 \quad \cos 2\theta = \cos 90^\circ = 0
\]
\[
\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]
\[
= 75 + 35 \times 0 + 28 \times 1 = 103 \text{ MPa}
\]
\[
\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]
\[
= -35 \times 1 + 28 \times 0
\]
\[
= -35 \text{ MPa}
\]
\[
\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y
\]
\[
\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}
\]
\[
= 110 + 40 - 103 = 47 \text{ MPa}
\]

**Example 7-2**

\[
\sigma_x = -46 \text{ MPa} \quad \sigma_y = 12 \text{ MPa}
\]
\[
\tau_{xy} = \tau_{yx} = -19 \text{ MPa}
\]

determine the stresses for \( \theta = -15^\circ \)

\[
\frac{\sigma_x + \sigma_y}{2} = -17 \text{ MPa} \quad \frac{\sigma_x - \sigma_y}{2} = -29 \text{ MPa}
\]
\[
\sin 2\theta = \sin (-30^\circ) = -0.5 \quad \cos 2\theta = \cos (-30^\circ) = 0.866
\]
\[
\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]
\[
= -17 + (-29) 0.866 + (-19) (-0.5) = -32.6 \text{ MPa}
\]
\[
\tau_{x_1y_1} = - \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

\[
= - ( -29)(-0.5) + (-19)0.866
\]

\[
= -31 \text{ MPa}
\]

\[
\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y
\]

\[
\sigma_{y_1} = \sigma_{x_1} + \sigma_y - \sigma_x
\]

\[
= -46 + 12 - (-32.6) = -1.4 \text{ MPa}
\]

### 7.3 Principal Stresses and Maximum Shear Stresses

\(\sigma_{x_1}\) and \(\tau_{x_1y_1}\) vary continuously as the element is rotated through the angle \(\theta\).

For design purposes, the largest positive and negative stresses are usually needed. The maximum and minimum normal stresses are called the principal stresses.

Consider the stress transformation equation

\[
\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

to find the maximum normal stress, we may set \(d \sigma_{x_1} / d \theta = 0\)

\[
\frac{d \sigma_{x_1}}{d \theta} = -(\sigma_x - \sigma_y) \sin 2\theta + 2 \tau_{xy} \cos 2\theta = 0
\]

We get
\[
\tan 2\theta_p = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}
\]

\(\theta_p\) defines the orientation of the principal plane, two values of \(2\theta_p\) from \(0 \sim 360^\circ\) and differ by \(180^\circ\)
\[ \therefore \theta_p \text{ has two values differ by } 90^\circ, \text{ we conclude that the principal stresses occur on mutually perpendicular plane} \]

also

\[ \cos 2\theta_p = \frac{(\sigma_x - \sigma_y) / 2}{R} \]

\[ \sin 2\theta_p = \frac{\tau_{xy}}{R} \]

where \( R = \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}} \)

substitute \( \cos 2\theta_p \) and \( \sin 2\theta_p \) into the expression of \( \sigma_{x1} \)

\[ \sigma_1 = (\sigma_{x1})_{max} = \frac{\sigma_x + \sigma_y}{2} + \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}} \]

and the smaller principal stress denoted by \( \sigma_2 \) is obtained

\[ \sigma_1 + \sigma_2 = \sigma_x + \sigma_y \]

\[ \sigma_2 = (\sigma_{x1})_{min} = \frac{\sigma_x + \sigma_y}{2} - \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}} \]

the principal stresses can be written as

\[ \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}} \]

+ sign gives the larger principal stress

- sign gives the smaller principal stress

\( \theta_{p_1} \) and \( \theta_{p_2} \) can be determined, but we cannot tell from the equation which angle is \( \theta_{p_1} \) and which is \( \theta_{p_2} \)

an important characteristic concerning the principal plane : the shear is
zero on the principal plane

\[ \tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \]

substitute \( 2\theta_p \) into this equation, we can get

\[ \tau_{x_1y_1} = 0 \]

for uniaxial and biaxial stress states, \( \tau_{xy} = 0 \)

\[ \tan 2\theta_p = 0 \quad \theta_p = 0^\circ \text{ and } 90^\circ \]

for pure shear stress, \( \sigma_x = \sigma_y = 0 \)

\[ \tan 2\theta_p = \infty \]
\[ \theta_p = 45^\circ \text{ and } 135^\circ \]

for the three-dimensional stress element, \( \sigma_z = 0 \) is also a principal stress, note that there are no shear stresses on the principal plane

Maximum Shear Stress

\[ \tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \]

\[ \frac{d \tau_{x_1y_1}}{d \theta} = - (\sigma_x - \sigma_y) \cos 2\theta - 2 \tau_{xy} \sin 2\theta = 0 \]

\[ \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2 \tau_{xy}} \]
$2\theta_s$ has two values between 0 and $360^\circ$

$\theta_s$ has two values between 0 and $180^\circ$, and differ by $90^\circ$

Comparing the angle $\theta_s$ and $\theta_p$, it is shown that

$$\tan 2\theta_s = -\frac{1}{\tan 2\theta_p} = -\cot 2\theta_p$$

i.e. $2\theta_s \perp 2\theta_p$, $2\theta_s = 2\theta_p \pm 90^\circ$

or $\theta_s = \theta_p \pm 45^\circ$

The plane of maximum shear stress occur at $45^\circ$ to the principal plane

Similarly we have

$$\cos 2\theta_s = \frac{\tau_{xy}}{R} \quad \sin 2\theta_s = -\frac{\sigma_x - \sigma_y}{2R}$$

And the corresponding maximum shear stress is

$$\tau_{max} = \left[ \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right]^{\frac{1}{2}}$$

The algebraically minimum shear stress $\tau_{min}$ has the same magnitude

A usually expression for the maximum shear stress can be obtained by $\sigma_1$ and $\sigma_2$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

The $\tau_{max}$ is equal to one half the difference of the principal stress

Normal stresses also act on the planes of maximum $\tau$, substituting $\theta_{si}$ in the formula of $\sigma_{xi}$, it is obtained
\[ \sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} = \sigma_{ave} = \sigma_{y_1} \]

\(\sigma_{ave}\) acts on both the plane of maximum and minimum \(\tau\) planes

if we make a three-dimensional analysis, we can establish that there are possible positions of element for maximum shear stress

\[(\tau_{max})_{x_1} = \pm \frac{\sigma_1}{2} \text{ rotate the element } 45^\circ \text{ about } x_1 \text{ axis} \]

\[(\tau_{max})_{y_1} = \pm \frac{\sigma_2}{2} \text{ rotate the element } 45^\circ \text{ about } y_1 \text{ axis} \]

\[(\tau_{max})_{z_1} = \pm \frac{\sigma_1 - \sigma_2}{2} \text{ rotate the element } 45^\circ \text{ about } z_1 \text{ axis} \]

if \(\sigma_1 > \sigma_2 > 0\), then \(\tau_{max} = \sigma_1 / 2 = (\tau_{max})_{y_1}\) for the element

Example 7-3

\(\sigma_x = 84 \text{ MPa} \quad \sigma_y = -30 \text{ MPa} \)

\(\tau_{xy} = -32 \text{ MPa} \)

determine the principal stresses and maximum shear stress and their directions
the principal angles $\theta_p$ can be obtained

$$\tan 2\theta_p = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-32)}{84 - (-30)} = -0.5614$$

$\theta_p = 150.6^\circ \text{ or } 330.6^\circ$

$\theta_p = 75.3^\circ \text{ or } 165.3^\circ$

$$\frac{\sigma_x + \sigma_y}{2} = \frac{(84 - 30)}{2} = 27 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{(84 + 30)}{2} = 57 \text{ MPa}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

for $2\theta_p = 330.6^\circ$ ($\theta_p = 165.3^\circ$) $\sigma_1 = 92.4 \text{ MPa}$

for $2\theta_p = 150.3^\circ$ ($\theta_p = 75.3^\circ$) $\sigma_2 = -38.4 \text{ MPa}$

check $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$ (O. K.)

alternative method for principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \left[ \frac{\sigma_x - \sigma_y}{2} \right]^2 + \tau_{xy}^2 \right]^{\frac{1}{2}}$$

$$= 27 \pm \left[ (57)^2 + (-32)^2 \right]^{\frac{1}{2}} = 27 \pm 65.4$$

thus $\sigma_1 = 92.4 \text{ MPa} \quad \sigma_2 = -38.4 \text{ MPa}$

$\theta_{p1} = 165.3^\circ \quad \theta_{p2} = 75.3^\circ$

the maximum shear stresses are given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 65.4 \text{ MPa}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ = 120.3^\circ$$
and \( \theta_{x2} = 120.2^\circ - 90^\circ = 30.3^\circ \)

and the normal stress acting on the planes of maximum shear stress are

\[
\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = 27 \text{ MPa}
\]

### 7.4 Mohr's Circle for Plane Stress

The transformation of plane stress can be represented in graphical form, known as Mohr's circle.

The equation of Mohr's circle can be derived from the transformation equations for plane stress

\[
\begin{align*}
\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2} &= \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\tau_{x1y1} &= -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\end{align*}
\]

To eliminate the parameter \( 2\theta \), we square both sides of each equation and then add together, it can be obtained

\[
(\sigma_{x1} - \frac{\sigma_x + \sigma_y}{2})^2 + \tau_{x1y1}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2
\]

Let \( \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \)

\[
R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2
\]

Then the above equation can be written

\[
(\sigma_{x1} - \sigma_{\text{ave}})^2 + \tau_{x1y1}^2 = R^2
\]
this is a equation of circle with $\sigma_{x1}$ and $\tau_{x1y1}$ as coordinates, the radius is $R$ and center at $\sigma_{x1} = \sigma_\text{ave}$, $\tau_{x1y1} = 0$

positive shear stress is plotted downward and a positive angle $2\theta$ is plotted counterclockwise

positive shear stress is plotted upward and a positive angle $2\theta$ is plotted clockwise

Construction of Mohr's Circle

(1) locate the center $C$ at $\sigma_{x1} = \sigma_\text{ave}$, $\tau_{x1y1} = 0$

(2) locate point $A$ which is at $\theta = 0$, $\sigma_{x1} = \sigma_x$, $\tau_{x1y1} = \tau_{xy}$

(3) locate point $B$ which is at $\theta = 90^\circ$, $\sigma_{x1} = \sigma_y$, $\tau_{x1y1} = -\tau_{xy}$

[Note that the line $AB$ must pass through point $C$]

(4) draw the circle through points $A$ and $B$ with center at $C$ this circle is the **Mohr's circle** with radius $R$

\[
R = \sqrt{\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}{2}}
\]

the stress state on an inclined element with an angle $\theta$ is represented at point $D$ on the Mohr's circle, which is measured an angle $2\theta$ counter-clockwise from point $A$
to show the coordinate at \( D \)

\[
\sigma_{x1} = \sigma_{ave} + R \cos (2\theta_p - 2\theta) \\
= \sigma_{ave} + R (\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta)
\]

\[\therefore \quad R \cos 2\theta_p = R \frac{\sigma_x - \sigma_y}{2R} = \frac{\sigma_x - \sigma_y}{2}
\]

\[R \sin 2\theta_p = R \frac{\tau_{xy}}{R} = \tau_{xy}
\]

\[\therefore \quad \sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta
\]

\[\tau_{xy} = R \sin (2\theta_p - 2\theta) = R (\sin 2\theta_p \cos 2\theta - \cos 2\theta_p \sin 2\theta)
\]

\[= \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\]

same results as the transformation equations

point \( D' \) represents the stress state on the face of the face 90° from the face represented by point \( D \), i.e. \( y_1 \) face
\[\begin{align*}
A (\sigma_x, \tau_{xy}) & \quad B (\sigma_y, -\tau_{xy}) & \quad C (\sigma_{\text{ave}}, 0) \\
D (\sigma_x, \tau_{x'yi}) & \quad D' (\sigma_y, -\tau_{x'yi})
\end{align*}\]

at point \( P_1 \) on the circle, \( \sigma_{x_1} = \sigma_{\text{max}} = \sigma_1 \)
hence, \( P_1 \) represents the stress state at principal plane
the other principal plane \( (\sigma_{\text{min}} = \sigma_2) \) is represented by \( P_2 \)

\[\begin{align*}
\sigma_1 &= OC + CP_1 = \frac{\sigma_x + \sigma_y}{2} + R \\
\sigma_2 &= OC - CP_2 = \frac{\sigma_x + \sigma_y}{2} - R
\end{align*}\]

the principal angle \( \theta_{p_1} \) can be obtained by

\[\begin{align*}
\cos 2\theta_{p_1} &= \frac{\sigma_x - \sigma_y}{2 R} \quad \text{or} \quad \sin 2\theta_{p_1} = \frac{\tau_{xy}}{R}
\end{align*}\]

and \( \theta_{p_2} = \theta_{p_1} + 90^\circ \)

comparing the Mohr's circle and the stress element, it is observed

Mohr's Circle : stress element
\[\begin{align*}
A & \quad \rightarrow \quad P_1 \ (2\theta_{p_1} \ \infty) & \quad x & \quad \rightarrow \quad x_1 \ (\theta_{p_1} \ \infty) \\
A & \quad \rightarrow \quad P_2 \ (2\theta_{p_1} + 180^\circ \ \infty) & \quad x & \quad \rightarrow \quad x_1 \ (\theta_{p_1} + 90^\circ \ \infty) \\
\text{or} & \quad (180^\circ - 2\theta_{p_1} \ \Psi) & \quad \text{or} & \quad (90^\circ - \theta_{p_1} \ \Psi)
\end{align*}\]

\[\begin{align*}
P_1 & \quad \rightarrow \quad S \ (90^\circ \ \Psi) & \quad x_1 & \quad \rightarrow \quad \tau_{\text{max}} \ (45^\circ \ \Psi)
\end{align*}\]

points \( S \) and \( S' \) representing the points of maximum and minimum shear stresses, are located on the circle at \( 90^\circ \) from points \( P_1 \) and \( P_2 \).
i.e. the planes of maximum and minimum shear stress are at 45° to the principal planes, and

\[ \tau_{\text{max}} = R = \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \]

if either \( \sigma_x \) or \( \sigma_y \) is negative, part of the circle will be located to the left of the origin.

Mohr's circle makes it possible to visualize the relationships between stresses acting planes at various angles, and it also serves as a simple memory device for obtaining the stress transformation equation.

Example 7-4

\[ \sigma_x = 90 \text{ MPa} \quad \sigma_y = 20 \text{ MPa} \]
\[ \tau_{xy} = 0 \quad \theta = 30^\circ \]

\[ \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{90 + 20}{2} = 55 \text{ MPa} \]

A (\( \theta = 0 \)) \( \sigma_{x1} = 90 \quad \tau_{x1y1} = 0 \)
B (\( \theta = 90^\circ \)) \( \sigma_{x1} = 20 \quad \tau_{x1y1} = 0 \)

\[ R = \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 35 \text{ MPa} \]

\( \theta = 30^\circ \quad 2\theta = 60^\circ \) (point D)

\[ \sigma_{x1} = \sigma_{\text{ave}} + R \cos 60^\circ \]
\[ = 55 + 35 \cos 60^\circ = 72.5 \text{ MPa} \]
\[ \tau_{x1y1} = -R \sin 60^\circ \]
\[ = -35 \sin 60^\circ = -30.3 \text{ MPa} \]
\[ \theta = 120^\circ \quad 2\theta = 240^\circ \quad \text{(point } D') \]
\[ \sigma_{x1} = \sigma_{ave} - R \cos 60^\circ \]
\[ = 55 - 35 \cos 60^\circ = 37.5 \text{ MPa} \]
\[ \tau_{x1y1} = R \sin 60^\circ \]
\[ = 35 \sin 60^\circ = 30.3 \text{ MPa} \]

Example 7-5
\[ \sigma_x = 100 \text{ MPa} \quad \sigma_y = 34 \text{ MPa} \]
\[ \tau_{xy} = 28 \text{ MPa} \]
determine the stresses on the face of \( \theta = 40^\circ \)
determine \( \sigma_1, \sigma_2 \) and \( \tau_{\max} \)

\[ \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{10 + 34}{2} = 67 \text{ MPa} \]
\[ A \quad (\theta = 0) \quad \sigma_{x1} = 100 \quad \tau_{x1y1} = 28 \]
\[ B \quad (\theta = 90^\circ) \quad \sigma_{x1} = 34 \quad \tau_{x1y1} = -28 \]

\[ R = \left[ \frac{\sigma_x - \sigma_y}{2} \right]^2 + \tau_{xy}^2 \]
\[ = 43 \text{ MPa} \]
\[ \theta = 40^\circ \quad 2\theta = 80^\circ \quad \text{(point } D) \]
\[ \tan \triangle ACP_1 = \frac{28}{33} = 0.848 \]
\[ \triangle ACP_1 = 2 \theta_{pi} = 40.3^\circ \]
\[ \triangle DCP_1 = 80^\circ - \triangle ACP_1 = 39.7^\circ \]
\[ \sigma_{x1} = \sigma_{ave} + R \cos 39.7^\circ = 100 \text{ MPa} \]
\( \tau_{x_1y_1} = -R \sin 39.7^\circ = -27.5 \text{ MPa} \)

Principal stresses are represented by \( P_1 \) and \( P_2 \)

\[
\begin{align*}
\sigma_1 &= \sigma_{\text{ave}} + R = 110 \text{ MPa} \\
2\theta_{p1} &= 40.3^\circ \quad \theta_{p1} = 20.15^\circ \\
\sigma_2 &= \sigma_{\text{ave}} - R = 24 \text{ MPa} \\
\theta_{p2} &= \theta_{p1} + 90^\circ = 110.15^\circ
\end{align*}
\]

Maximum shear stress

\[
\tau_{\text{max}} = R = 43 \text{ MPa}
\]

\[
\theta_3 = \theta_{p1} - 45^\circ = -24.85^\circ
\]

\[
\sigma_{x_1} = \sigma_{\text{ave}} = 67 \text{ MPa}
\]

Example 7-6

\[
\begin{align*}
\sigma_x &= -50 \text{ MPa} \quad \sigma_y = 10 \text{ MPa} \\
\tau_{xy} &= -40 \text{ MPa} \quad \theta = 30^\circ
\end{align*}
\]

determine \( \sigma_{x_1}, \tau_{x_1y_1} \) on \( \theta = 45^\circ \)

determine \( \sigma_1, \sigma_2 \) and \( \tau_{\text{max}} \)

\[
\sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = \frac{-50 + 10}{2} = -20 \text{ MPa}
\]

\[
\begin{align*}
A (\theta = 0) \quad \sigma_{x_1} &= -50 \quad \tau_{x_1y_1} = -40 \\
B (\theta = 90^\circ) \quad \sigma_{x_1} &= 10 \quad \tau_{x_1y_1} = 40
\end{align*}
\]

\[
R = \left[ \frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2 \right]^{1/2} = 50 \text{ MPa}
\]
\[ \theta = 45^\circ \quad 2\theta = 90^\circ \text{ (point } D) \]
\[ \tan \angle ACP_2 = \frac{40}{30} = 1.333 \]
\[ \angle ACP_2 = 2\theta_{p_2} = 53.13^\circ \]
\[ \angle DCP_2 = 90^\circ - \angle ACP_2 = 36.87^\circ \]
\[ \sigma_{x_1} = -20 - 50 \cos 36.87^\circ = -60 \text{ MPa} \]
\[ \tau_{x_1y_1} = 50 \sin 36.87^\circ = 30 \text{ MPa} \]

at \( D' \)
\[ \sigma_{x_1} = -50 + 10 - (-60) = 20 \text{ MPa} \]
\[ \tau_{x_1y_1} = -30 \text{ MPa} \]

principal stresses are represented by \( P_1 \) and \( P_2 \)
\[ \sigma_1 = \sigma_{ave} + R = 30 \text{ MPa} \]
\[ 2\theta_{p_1} = 2\theta_{p_2} + 180^\circ = 233.13^\circ \quad \theta_{p_1} = 116.6^\circ \]
\[ \sigma_2 = \sigma_{ave} - R = -70 \text{ MPa} \quad \theta_{p_2} = 26.6^\circ \]

maximum shear stress
\[ \tau_{max} = R = 50 \text{ MPa} \]
\[ \theta_s = \theta_{p_1} - 45^\circ = 71.6^\circ \]
\[ \sigma_{x_1} = \sigma_{ave} = -20 \text{ MPa} \]
7.5 Hook's Law for Plane Stress

for the plane stress with normal stresses $\sigma_x$ and $\sigma_y$, the normal strains are

$$
\varepsilon_x = \frac{(\sigma_x - \nu \sigma_y)}{E}
$$
$$
\varepsilon_y = \frac{(\sigma_y - \nu \sigma_x)}{E}
$$
$$
\varepsilon_z = -\nu (\sigma_x + \sigma_y) / E
$$

the first two equations can be solved for the stresses in terms of strains

$$
\sigma_x = E \left( \varepsilon_x + \nu \varepsilon_y \right) / (1 - \nu^2)
$$
$$
\sigma_y = E \left( \varepsilon_y + \nu \varepsilon_x \right) / (1 - \nu^2)
$$

for the pure shear stress $\tau_{xy}$, the shear strain $\gamma_{xy}$ is

$$
\gamma_{xy} = \frac{\tau_{xy}}{G}
$$
or

$$
\tau_{xy} = G \gamma_{xy}
$$

the three material parameters with the relation

$$
G = E / [2 (1 + \nu)]
$$

volume change

$$
V_0 = a \ b \ c
$$
$$
V_1 = a \left( 1 + \varepsilon_x \right) b \left( 1 + \varepsilon_y \right) c \left( 1 + \varepsilon_z \right)
$$
$$
= V_0 \left( 1 + \varepsilon_x \right) \left( 1 + \varepsilon_y \right) \left( 1 + \varepsilon_z \right)
$$
$$
= V_0 \left( 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_x \varepsilon_z + \varepsilon_x \varepsilon_y \varepsilon_z \right)
$$
$$
\approx V_0 \left( 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z \right)
$$

and the volume change is
\[ \Delta V = V_1 - V_0 = V_0 (\varepsilon_x + \varepsilon_y + \varepsilon_z) \]

the unit volume change or dilatation \( e \) is defined

\[ e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z \]

for uniaxial stress \( \sigma_x \) only

\[ e = \frac{\Delta V}{V_0} = \sigma_x (1 - 2 \nu) / E \]

for plane stress \( \sigma_x \) and \( \sigma_y \)

\[ e = \frac{\Delta V}{V_0} = (\sigma_x + \sigma_y)(1 - 2 \nu) / E \]

Strain-energy density in plane stress

\[ u = \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) \]

\[ = \left( \sigma_x^2 + \sigma_y^2 - 2 \nu \sigma_x \sigma_y \right) / 2E + \frac{\tau_{xy}^2}{2G} \]

\[ = E \left( \varepsilon_x^2 + \varepsilon_y^2 + 2 \nu \varepsilon_x \varepsilon_y \right) / \left[ 2(1 - \nu^2) \right] + G \gamma_{xy}^2 / 2 \]

7.6 Triaxial Stress

a stress element subjected to normal stress \( \sigma_x, \sigma_y \) and \( \sigma_z \) is said to be in a triaxial stress state

on the inclined plane parallel to the \( z \) axis, only \( \sigma \) and \( \tau \) on this plane, the maximum shear stress occurs on the plane by a 45\(^\circ\) rotation about \( z \) axis is

\[ (\tau_{\max})_z = \pm \frac{\sigma_x - \sigma_y}{2} \]

similarly
\[
(t_{\text{max}})_x = \pm \frac{\sigma_y - \sigma_z}{2} \\
(t_{\text{max}})_y = \pm \frac{\sigma_x - \sigma_z}{2}
\]

the absolute maximum shear stress is the difference between algebraically largest and smallest of the three principle stresses

the Mohr’s circles for a 3-D element is shown

Hooke’s law for triaxial stress, the normal strains are

\[
e_x = \frac{\sigma_x}{E} - \nu (\sigma_y + \sigma_z) / E \\
e_y = \frac{\sigma_y}{E} - \nu (\sigma_x + \sigma_z) / E \\
e_z = \frac{\sigma_z}{E} - \nu (\sigma_x + \sigma_y) / E
\]

stresses in terms of strains are

\[
\sigma_x = \frac{[E (1 - \nu) e_x + \nu (e_y + e_z)]}{(1 + \nu) (1 - 2\nu)} \\
\sigma_y = \frac{[E (1 - \nu) e_y + \nu (e_x + e_z)]}{(1 + \nu) (1 - 2\nu)} \\
\sigma_z = \frac{[E (1 - \nu) e_z + \nu (e_x + e_y)]}{(1 + \nu) (1 - 2\nu)}
\]

in the special case of biaxial stress, \(\sigma_z = 0\), the result are the same as in section 7.5

the unit volume change is also obtained

\[
e = \frac{\Delta V}{V_0} = e_x + e_y + e_z \\
= (\sigma_x + \sigma_y + \sigma_z) (1 - 2\nu) / E
\]

and the strain energy density is

\[
u = \frac{1}{2} \left( \sigma_x e_x + \sigma_y e_y + \sigma_z e_z \right) \\
= \frac{(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)}{2E} - \nu \left( \sigma_x \sigma_y + \sigma_x \sigma_z + \sigma_y \sigma_z \right) / E
\]
\[
E \left[ (1 - v)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2v (\varepsilon_x\varepsilon_y + \varepsilon_x\varepsilon_z + \varepsilon_y\varepsilon_z) \right] = \frac{2(1 + v)(1 - 2v)}{2}
\]

for spherical stress

\[
\sigma_x = \sigma_y = \sigma_z = \sigma_0
\]

then the normal strains are

\[
\varepsilon_0 = \sigma_0 (1 - 2v) / E
\]

and the unit volume change is

\[
e = 3 \varepsilon_0 = 3\sigma_0 (1 - 2v) / E
\]

define the bulk modulus of elasticity as

\[
K = E / 3 (1 - 2v)
\]

then \( e \) may expressed as \( e = \sigma_0 / K \)

and the bulk modulus is \( K = \sigma_0 / e \)

for an object submerged in water, the stress is spherical state, it is often called hydrostatic stress

7.7 Plane Strain

the normal and shear strains at a point in body vary with direction, for plane strain, the strain components are

\[
\varepsilon_x, \varepsilon_y, \gamma_{xy} \quad \text{and} \quad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0
\]

plane stress : \( \sigma_z = \tau_{xz} = \tau_{yz} = 0 \)

but \( \varepsilon_z \neq 0 \)

plane strain : \( \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \)

but \( \sigma_z \neq 0 \)
plane stress and plane strain do not occur simultaneously in general
special cases for plane stress  ==>  plane strain

1. $\sigma_x = -\sigma_y$, $\varepsilon_z = -\nu (\sigma_x + \sigma_y) / E = 0$
2. $\nu = 0$, $\varepsilon_z = 0$ for $\sigma_z = 0$

we will derive the strain transformation equations for the case of plane
strain, the equations actually are valid even when $\varepsilon_z$ exists

assume $\varepsilon_x$, $\varepsilon_y$, $\gamma_{xy}$ associated with
$x$ and $y$ axes are known, to determine
$\varepsilon_{x1}$, $\varepsilon_{y1}$, $\gamma_{x1y1}$ associated with $x_1$ and
$y_1$ axes where are rotated counterclockwise
through an angle $\theta$ form $x$ and $y$
axes.

consider first the strain $\varepsilon_x$ in $x$ direction

$$\delta x = \varepsilon_x \, dx$$

$$\delta x_1 = \varepsilon_x \, dx \cos \theta$$

similarly for $\varepsilon_y$ in $y$ direction

$$\delta y = \varepsilon_y \, dy$$

$$\delta x_1 = \varepsilon_y \, dy \sin \theta$$

consider shear strain $\gamma_{xy}$ in $xy$ plane

$$\delta x_1 = \gamma_{xy} \, dy \cos \theta$$

then the total increase $\Delta d$ in $x_1$ direction is

$$\Delta d = \varepsilon_x \, dx \cos \theta + \varepsilon_y \, dy \sin \theta + \gamma_{xy} \, dy \cos \theta$$
and the strain in $x_1$ direction is

$$
\varepsilon_{x_1} = \frac{\Delta d}{ds} = \varepsilon_x \frac{dx}{ds} \cos \theta + \varepsilon_y \frac{dy}{ds} \sin \theta + \gamma_{xy} \frac{dy}{ds} \cos \theta
$$

but $\frac{dx}{ds} = \cos \theta$ and $\frac{dy}{ds} = \sin \theta$

thus $\varepsilon_{x_1} = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$

substituting $\theta + 90^\circ$ for $\theta$, the $\varepsilon_{y_1}$ is obtained

$$
\varepsilon_{y_1} = \varepsilon_x \sin^2 \theta + \varepsilon_y \cos^2 \theta - \gamma_{xy} \sin \theta \cos \theta
$$

hence $\varepsilon_{x_1} + \varepsilon_{y_1} = \varepsilon_x + \varepsilon_y$

To obtain the shear strain $\gamma'_{x_1y_1}$, this strain is equal to the decrease in angle between lines that were initially along $x_1$ and $y_1$ axes

$$
\gamma'_{x_1y_1} = a + \beta
$$

$$
a = -a_1 + a_2 - a_3
$$

$$
= -\frac{dx}{ds} \sin \theta + \frac{dy}{ds} \cos \theta - \gamma_{xy} \frac{dy}{ds} \sin \theta
$$

$$
= -(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta - \gamma_{xy} \sin^2 \theta
$$

similarly

$$
\beta = -(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta + \gamma_{xy} \cos^2 \theta
$$

then the shear strain $\gamma'_{x_1y_1}$ is

$$
\gamma'_{x_1y_1} = -2(\varepsilon_x - \varepsilon_y) \sin \theta \cos \theta + \gamma_{xy} (\cos^2 \theta - \sin^2 \theta)
$$

Use some trigonometric identities, the transformation equations for plane strain are
\[
e_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta
\]

\[
\frac{\gamma'_{x1y1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta
\]

the equations are the counterparts of plane stress

<table>
<thead>
<tr>
<th>stresses</th>
<th>strains</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma_x)</td>
<td>(\varepsilon_x)</td>
</tr>
<tr>
<td>(\sigma_y)</td>
<td>(\varepsilon_y)</td>
</tr>
<tr>
<td>(\tau_{xy})</td>
<td>(\gamma_{xy}/2)</td>
</tr>
<tr>
<td>(\sigma_{x1})</td>
<td>(\varepsilon_{x1})</td>
</tr>
<tr>
<td>(\sigma_{y1})</td>
<td>(\varepsilon_{y1})</td>
</tr>
<tr>
<td>(\tau_{x1y1})</td>
<td>(\gamma'_{x1y1}/2)</td>
</tr>
</tbody>
</table>

principal strains exist on perpendicular planes with angles \(\theta_p\)

\[
\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}
\]

the principal strains can be calculated

\[
e_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \left[ \left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma_{xy}}{2} \right)^2 \right]^{1/2}
\]

the maximum shear strains exists at 45° to the direction of the principal strains

\[
\frac{\gamma_{max}}{2} = \left[ \left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma'_{xy}}{2} \right)^2 \right]^{1/2}
\]

and the normal strains in the directions of maximum shear strains are

\[
e_{max} = \frac{\varepsilon_x + \varepsilon_y}{2}
\]
the principal strains and principal stresses occur in the same directions

Mohr's Circle for Plane Strain

plane strains at a point can be measured by strain rosette, then the stresses at this point can be calculated, also the principal strains and principal stresses can be obtained

Example 7-7

\[
\varepsilon_x = 340 \times 10^{-6}, \quad \varepsilon_y = 110 \times 10^{-6}, \\
\gamma_{xy} = 180 \times 10^{-6}
\]

determine the strains for \( \theta = 30^\circ \), principal strains, maximum shear strain

\[
\frac{\varepsilon_x + \varepsilon_y}{2} = \frac{(340 + 110) \times 10^{-6}}{2} = 225 \times 10^{-6}
\]
\[
\frac{\varepsilon_x - \varepsilon_y}{2} = \frac{(340 - 110) \times 10^{-6}}{2} = 115 \times 10^{-6}
\]

\[
\gamma_{xy} / 2 = 90 \times 10^{-6}
\]

\[
\theta = 30^\circ \quad 2\theta = 60^\circ
\]

\[
\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta
\]

\[
= 225 \times 10^{-6} + (115 \times 10^{-6}) \cos 60^\circ + (90 \times 10^{-6}) \sin 60^\circ
\]

\[
= 360 \times 10^{-6}
\]

\[
\frac{\gamma'_{x1y1}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta
\]

\[
= -55 \times 10^{-6}
\]

\[
\gamma'_{x1y1} = -110 \times 10^{-6}
\]

\[
\varepsilon_{y1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x1}
\]

\[
= 90 \times 10^{-6}
\]

\[
\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \left[ \frac{\varepsilon_x - \varepsilon_y}{2} + \frac{\gamma_{xy}}{2} \right]^{\frac{1}{2}}
\]

\[
= 225 \times 10^{-6} \pm \left[ (115 \times 10^{-6})^2 + (90 \times 10^{-6})^2 \right]^{\frac{1}{2}}
\]

\[
= (225 \pm 146) \times 10^{-6}
\]

\[
\varepsilon_1 = 370 \times 10^{-6} \quad \varepsilon_2 = 80 \times 10^{-6}
\]

the angles of principal directions are

\[
\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{180}{340 - 110} = 0.7826
\]

\[
2\theta_p = 38^\circ \quad \text{and} \quad 218^\circ
\]
\[
\begin{align*}
\theta_p & = 19^\circ \quad \text{and} \quad 109^\circ \\
\varepsilon_1 & = 370 \times 10^{-6} \quad \theta_p = 19^\circ \\
\varepsilon_2 & = 80 \times 10^{-6} \quad \theta_p = 109^\circ \\
\end{align*}
\]

Note that \( \varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y \)

The maximum shear strain is

\[
\gamma_{\text{max}} = \frac{1}{2} \left[ \left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \frac{\gamma_{xy}^2}{2} \right] = 146 \times 10^{-6}
\]

\[
\gamma_{\text{max}} = 292 \times 10^{-6}
\]

\[
\theta_{s_1} = \theta_p - 45^\circ = -26^\circ
\]

\[
\theta_{s_2} = \theta_{s_1} + 90^\circ = 64^\circ
\]

The normal strains at this direction is

\[
\varepsilon_{\text{ave}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 225 \times 10^{-6}
\]

All of this results can be obtained from Mohr's circle.
Example 7-8

the plane strains measured by a 45°
strain rosette are \( \varepsilon_a, \varepsilon_b \) and \( \varepsilon_c \)
determine \( \varepsilon_x, \varepsilon_y \) and \( \gamma_{xy} \)

\[
\begin{align*}
\varepsilon_x &= \varepsilon_a \\
\varepsilon_y &= \varepsilon_b \\
\varepsilon_{x1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \\
\varepsilon_b &= \frac{\varepsilon_a + \varepsilon_c}{2} + \frac{\varepsilon_a - \varepsilon_c}{2} \cos 90^\circ + \frac{\gamma_{xy}}{2} \\
\gamma_{xy} &= 2 \varepsilon_b - \varepsilon_a - \varepsilon_c
\end{align*}
\]

for \( \theta = 45^\circ \), \( \varepsilon_{x1} = \varepsilon_b \)

solve for \( \gamma_{xy} \), we get

\[ \gamma_{xy} = 2 \varepsilon_b - \varepsilon_a - \varepsilon_c \]

the strains \( \varepsilon_x, \varepsilon_y \) and \( \gamma_{xy} \) can be determined from the
strain-gage reading
also the strains \( \varepsilon_{x1}, \varepsilon_{y1} \) and \( \gamma_{x1y1} \) can be calculate at any angle \( \theta \)