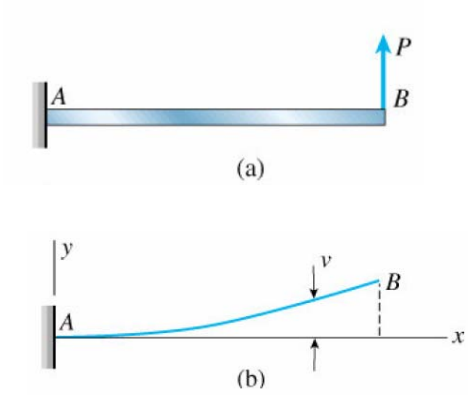
Chapter 5 Stresses in Beam (Basic Topics)

5.1 Introduction

Beam : loads acting transversely to the longitudinal axis



(a)
(b)

the loads create shear forces and bending moments, stresses and strains due to V and M are discussed in this chapter

lateral loads acting on a beam cause the beam to bend, thereby deforming the axis of the beam into curve line, this is known as the **deflection curve** of the beam

the beams are assumed to be symmetric about x-y plane, i.e. y-axis is an axis of symmetric of the cross section, all loads are assumed to act in the x-y plane, then the bending deflection occurs in the same plane, it is known as the **plane of bending**

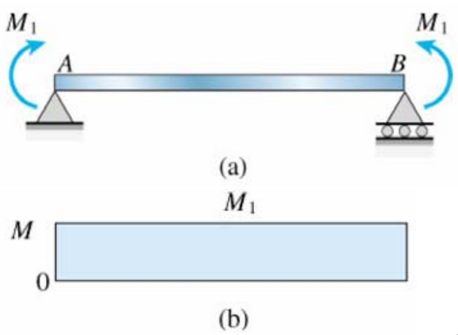
the **deflection** of the beam is the displacement of that point from its original position, measured in y direction

5.2 Pure Bending and Nonuniform Bending

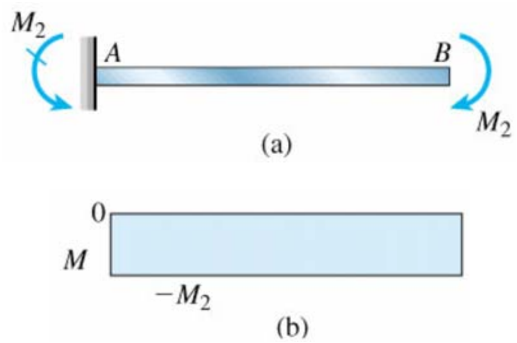
pure bending :

$$M = \text{constant} \qquad V = dM / dx = 0$$

pure bending in simple beam and cantilever beam are shown



(a)
(b)

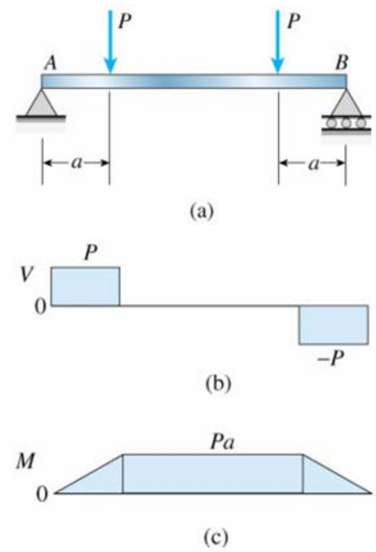


(a)
(b)

nonuniform bending :

$M \neq$ constant

$V = dM/dx \neq 0$

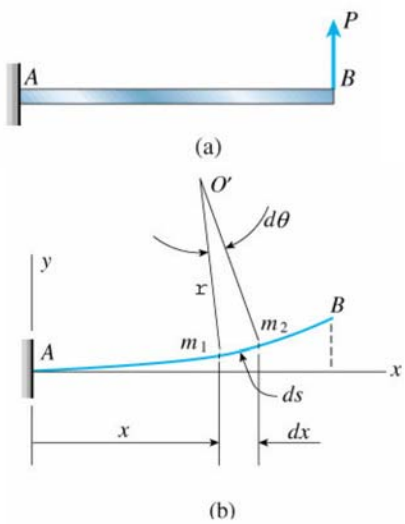


(a)
(b)
(c)

simple beam with central region in pure bending and end regions in nonuniform bending is shown

5.3 Curvature of a Beam

consider a cantilever beam subjected to a load P



(a)
(b)

choose 2 points m_1 and m_2 on the deflection curve, their normals intersect at point O', is called the center of curvature, the distance m_1O' is called radius of curvature ρ, and the curvature κ is defined as

$$\kappa = 1/\rho$$

and we have $\rho\, d\theta = ds$

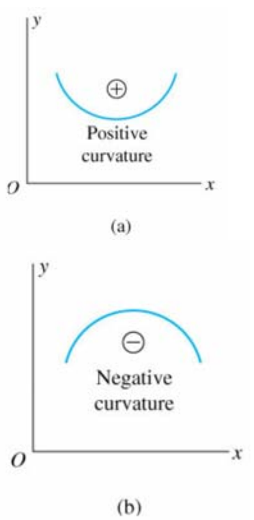
if the deflection is small $ds \simeq dx$, then

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx}$$

sign convention for curvature

+ : beam is bent concave upward (convex downward)

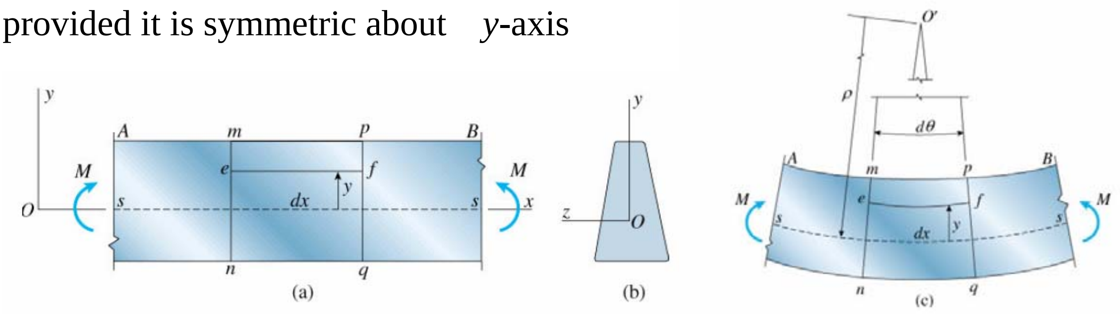
- : beam is bent concave downward (convex upward)



(a)
(b)

5.4 Longitudinal Strains in Beams

consider a portion ab of a beam in pure bending produced by a positive bending moment M, the cross section may be of any shape provided it is symmetric about y-axis



under the moment M, its axis is bent into a circular curve, cross section mn and pq remain plane and normal to longitudinal lines (plane remains plane can be established by experimental result)

$\because$ the symmetry of the beam and loading, it requires that all elements of the beam deform in an identical manner ($\therefore$ the curve is circular), this are valid for any material (elastic or inelastic)

due to bending deformation, cross sections mn and pq rotate w.r.t. each other about axes perpendicular to the xy plane

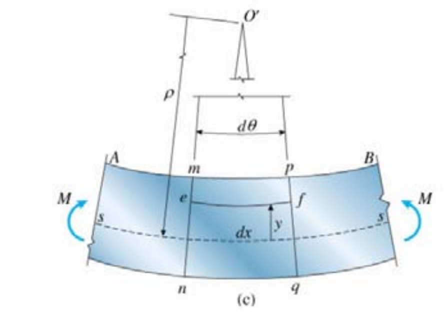
longitudinal lines on the convex (lower) side (nq) are elongated, and on the concave (upper) side (mp) are shortened

the surface ss in which longitudinal lines do not change in length is called the **neutral surface**, its intersection with the cross-sectional plane is called neutral axis, for instance, the z axis is the neutral axis of the cross section

in the deformed element, denote ρ the distance from O' to N.S. (or N.A.), thus

$$\rho \, d\theta \quad = \quad dx$$

consider the longitudinal line ef, the length L_1 after bending is



$$L_1 = (\rho - y)\,d\theta = dx - \frac{y}{\rho}\,dx$$

then $\triangle_{ef} = L_1 - dx = -\frac{y}{\rho}\,dx$

and the strain of line ef is

$$\varepsilon_x = \frac{\triangle_{ef}}{dx} = -\frac{y}{\rho} = -\kappa\,y$$

ε_x vary linear with y (the distance from N.S.)

$y > 0$ (above N. S.) $\varepsilon = -$

$y < 0$ (below N. S.) $\varepsilon = +$

the longitudinal strains in a beam are accompanied by transverse strains in the y and z directions because of the effects of Poisson's ratio

Example 5-1

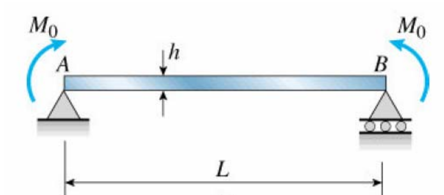
a simply supported beam AB,

$L = 4.9$ m $h = 300$ mm

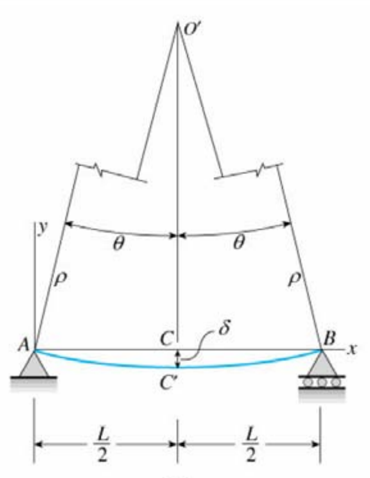
bent by M_0 into a circular arc

$\varepsilon_{bottom} = \varepsilon_x = 0.00125$

determine ρ, κ, and δ (midpoint deflection)



(a)



(b)

$$\rho = -\frac{y}{\varepsilon_x} = -\frac{-150}{0.00125}$$

$$= 120 \text{ m}$$

$$\kappa = \frac{1}{\rho} = 8.33 \times 10^{-3}\ \text{m}^{-1}$$

$$\delta = \rho\,(1 - \cos\theta)$$

$\because$ ρ is large, $\therefore$ the deflection curve is very flat

then $$\sin\theta = \frac{L/2}{\rho} = \frac{8 \times 12}{2 \times 2{,}400} = 0.020$$

$$\theta = 0.02 \text{ rad} = 1.146^{\circ}$$

then $$\delta = 120 \times 10^{3}\,(1 - \cos 1.146^{\circ}) = 24 \text{ mm}$$

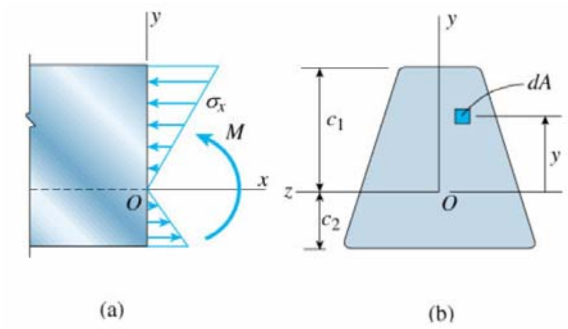
5.4 Normal Stress in Beams (Linear Elastic Materials)

$\because$ ε_x occurs due to bending, $\therefore$ the longitudinal line of the beam is subjected only to tension or compression, if the material is linear elastic

then $$\sigma_x = E\,\varepsilon_x = -E\,\kappa\,y$$

σ vary linear with distance y from the neutral surface

consider a positive bending moment M applied, stresses are positive below N.S. and negative above N.S.



$\because$ no axial force acts on the cross section, the only resultant is M, thus two equations must satisfy for static equilibrium condition

i.e. $$\Sigma F_x = \int \sigma\, dA = -\int E\,\kappa\,y\, dA = 0$$

$\because$ E and κ are constants at the cross section, thus we have

$$\int y\, dA = 0$$

we conclude that the neutral axis passes through the centroid of the cross section, also for the symmetrical condition in y axis, the y axis must pass through the centroid, hence, the origin of coordinates O is located at the centroid of the cross section

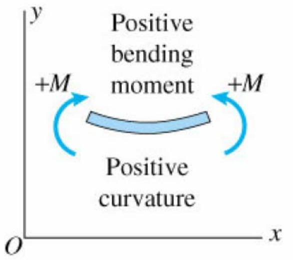
the moment resultant of stress σ_x is

$$dM = - \sigma_x\, y\, dA$$

then $\quad M = - \int \sigma_x\, y\, dA = \int E\, \kappa\, y^2\, dA = E\, \kappa \int y^2\, dA$

$$M = E\, \kappa\, I$$

where $I = \int y^2\, dA$ is the moment of inertia of the cross-sectional area w. r. t. z axis

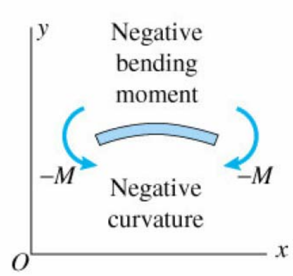


thus $\quad \kappa = \dfrac{1}{\rho} = \dfrac{M}{E\, I}$

this is the **moment-curvature equation**, and EI is called flexural rigidity

$+ M \;\Rightarrow\; +$ curvature

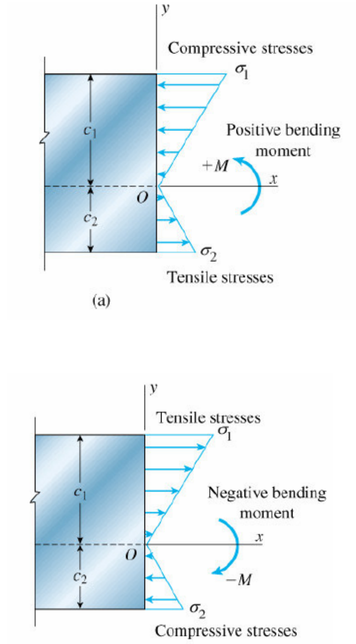
$- M \;\Rightarrow\; -$ curvature



the normal stress is

$$\sigma_x = - E\, \kappa\, y = - E\, y\, (\frac{M}{E\, I}) = - \frac{M\, y}{I}$$

this is called the **flexure formula**, the stress σ_x is called bending stresses or flexural stresses



(a) (b)

σ_x vary linearly with y

$\sigma_x \simeq M \qquad \sigma_x \simeq 1/I$

the maximum tensile and compressive stresses occur at the points located farthest from the N.A.

$$\sigma_1 = -\frac{M c_1}{I} = -\frac{M}{S_1}$$

$$\sigma_2 = \frac{M c_2}{I} = \frac{M}{S_2}$$

where $S_1 = \dfrac{I}{c_1}$, $S_2 = \dfrac{I}{c_2}$ are known as the section moduli

if the cross section is symmetric w.r.t. z axis (double symmetric cross section), then $c_1 = c_2 = c$

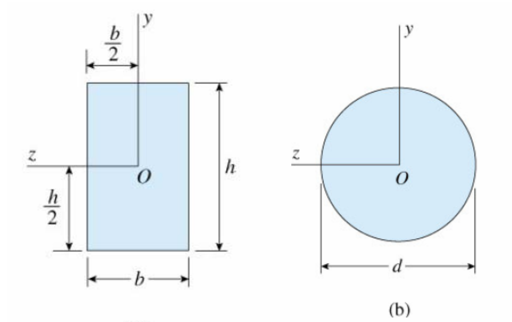
thus $S_1 = S_2$ and $\sigma_1 = -\sigma_2 = -\dfrac{M c}{I} = -\dfrac{M}{S}$

for rectangular cross section

$$I = \frac{b h^3}{12} \qquad S = \frac{b h^2}{6}$$

for circular cross section

$$I = \frac{\pi d^4}{64} \qquad S = \frac{\pi d^3}{32}$$



(a) (b)

the preceding analysis of normal stress in beams concerned pure bending, no shear force

in the case of nonuniform bending ($V \neq 0$), shear force produces warping (out of plane distortion), plane section no longer remain plane after bending, but the normal stress σ_x calculated from the flexure formula are not significantly altered by the presence of shear force and warping

we may justifiably use the theory of pure bending for calculating σ_x even when we have nonuniform bending

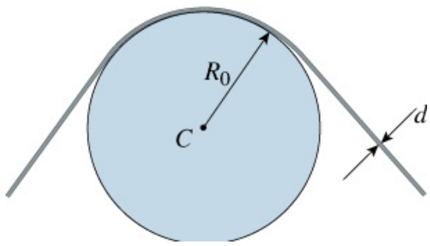
the flexure formula gives results in the beam where the stress distribution is not disrupted by irregularities in the shape, or by discontinuous in loading (otherwise, stress concentration occurs)

example 5-2

a steel wire of diameter $d = 4$ mm is bent around a cylindrical drum of radius $R_0 = 0.5$ m

$E = 200$ GPa $\sigma_{pl} = 1200$ MPa

determine M and σ_{max}



the radius of curvature of the wire is

$$\rho = R_0 + \frac{d}{2}$$

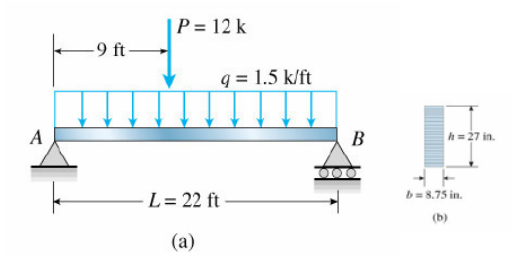
$$M = \frac{EI}{\rho} = \frac{2\,EI}{2\,R_0 + d} = \frac{\pi\,E\,d^4}{32(2R_0 + d)}$$

$$= \frac{\pi\,(200 \text{ x } 10^3)\,4^4}{32\,(2 \text{ x } 500 + 4)} = 5007 \text{ N-mm} = 5.007 \text{ N-m}$$

$$\sigma_{max} = \frac{M}{S} = \frac{M}{I\,/\,(d/2)} = \frac{M\,d}{2\,I} = \frac{2\,EI\,d}{2\,I\,(2\,R_0 + d)} = \frac{E\,d}{2\,R_0 + d}$$

$$= \frac{200 \times 10^3 \times 4}{2 \times 500 + 4} = 796.8 \text{ MPa} \quad < \quad 1{,}200 \text{ MPa (OK)}$$

Example 5-3



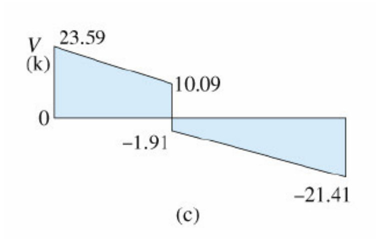
(a)

a simple beam *AB* of length L = 6.7 m

q = 22 kN/m $\qquad$ P = 50 kN

b = 220 mm $\qquad$ h = 700 mm

determine the maximum tensile and compressive stresses due to bending

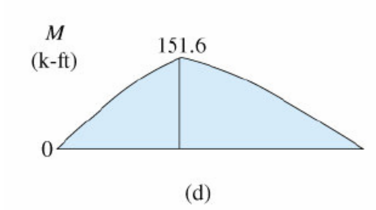


(c)

firstly, construct the *V-dia* and *M-dia*

σ_{max} occurs at the section of M_{max}

$$M_{max} = 193.9 \text{ kN-m}$$



(d)

the section modulus S of the section is

$$S = \frac{b\,h^2}{6} = \frac{0.22 \times 0.7^2}{6} = 0.018 \text{ m}^3$$

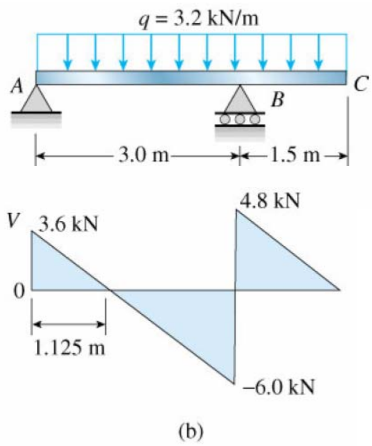
$$\sigma_t = \sigma_2 = \frac{M}{S} = \frac{139.9 \text{ kN-m}}{0.018 \text{ m}^3} = 10.8 \text{ MPa}$$

$$\sigma_c = \sigma_1 = -\frac{M}{S} = -10.8 \text{ MPa}$$

Example 5-4

an overhanged beam *ABC* subjected uniform load of intensity q = 3.2 kN/m

for the cross section (channel section)



(b)

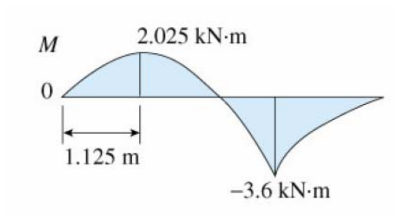
t = 12 mm b = 300 mm h = 80 mm

determine the maximum tensile and compressive stresses in the beam

construct the *V-dia.* and *M-dia.* first

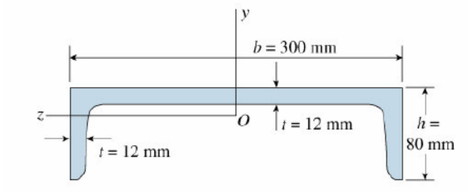
we can find $+M_{max} = 2.205$ kN-m

$-M_{max} = -3.6$ kN-m



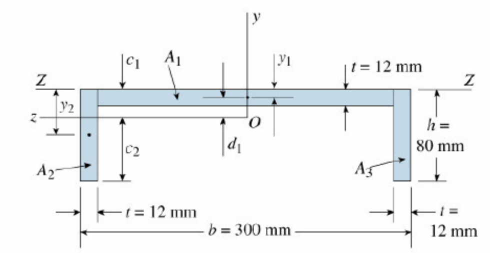
(c)

next, we want to find the N. A. of the section



(a)

	A(mm^2)	y(mm)	$A\,y$ (mm^3)
A_1	3,312	6	19,872
A_2	960	40	38,400
A_3	960	40	38,400
total	5,232		96,672



(b)

$$c_1 = \frac{\Sigma A_i\, y_i}{\Sigma A_i} = \frac{96{,}672}{5{,}232} = 18.48 \text{ mm}$$

$$c_2 = h - c_1 = 61.52 \text{ mm}$$

moment of inertia of the section is

$$I_{z1} = I_{zc} + A_1\, d_1^2$$

$$I_{zc} = \frac{1}{12}(b - 2t)\, t^3 = \frac{1}{12}\, 276 \times 12^3 = 39744 \text{ mm}^4$$

$$d_1 = c_1 - t/2 = 12.48 \text{ mm}$$

$$I_{z1} = 39{,}744 + 3{,}312 \times 12.48^2 = 555{,}600 \text{ mm}^4$$

similarly $I_{z2} = I_{z3} = 956{,}000$ mm^4

then the centroidal moment of inertia I_z is

$$I_z = I_{z1} + I_{z2} + I_{z3} = 2.469 \times 10^6 \text{ mm}^4$$

$$S_1 = \frac{I_z}{c_1} = 133{,}600 \text{ mm}^3 \qquad S_2 = \frac{I_z}{c_2} = 40{,}100 \text{ mm}^3$$

at the section of maximum positive moment

$$\sigma_t = \sigma_2 = \frac{M}{S_2} = \frac{2.025 \times 10^3 \times 10^3}{40{,}100} = 50.5 \text{ MPa}$$

$$\sigma_c = \sigma_1 = -\frac{M}{S_1} = -\frac{2.025 \times 10^3 \times 10^3}{133{,}600} = -15.2 \text{ MPa}$$

at the section of maximum negative moment

$$\sigma_t = \sigma_1 = -\frac{M}{S_1} = -\frac{-3.6 \times 10^3 \times 10^3}{133{,}600} = 26.9 \text{ MPa}$$

$$\sigma_c = \sigma_2 = \frac{M}{S_2} = -\frac{-3.6 \times 10^3 \times 10^3}{40{,}100} = -89.8 \text{ MPa}$$

thus $(\sigma_t)_{max}$ occurs at the section of maximum positive moment

$$(\sigma_t)_{max} = 50.5 \text{ MPa}$$

and $(\sigma_c)_{max}$ occurs at the section of maximum negative moment

$$(\sigma_c)_{max} = -89.8 \text{ MPa}$$

5.6 Design of Beams for Bending Stresses

design a beam : type of construction, materials, loads and environmental conditions beam shape and size : actual stresses do not exceed the allowable stress

for the bending stress, the section modulus S must be larger than M / σ

i.e. $S_{min} = M_{max} / \sigma_{allow}$

σ_{allow} is based upon the properties of the material and magnitude of the desired factor of safety

if σ_{allow} are the same for tension and compression, doubly symmetric section is logical to choose

if σ_{allow} are different for tension and compression, unsymmetric cross section such that the distance to the extreme fibers are in nearly the same ratio as the respective allowable stresses

select a beam not only the required S, but also the smallest cross-sectional area

Beam of Standardized Shapes and Sizes

steel, aluminum and wood beams are manufactured in standard sizes

steel : American Institute of Steel Construction (AISC)

Eurocode

e.g. wide-flange section W 30 x 211 depth = 30 in, 211 lb/ft

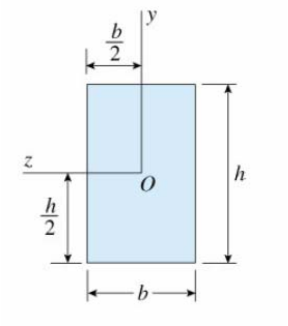
HE 1000 B depth = 1000 mm, 314 kgf/m etc

other sections : S shape (I beam), C shape (channel section)

L shape (angle section)

aluminum beams can be extruded in almost any desired shape since the dies are relatively easy to make

wood beam always made in rectangular cross section, such as 4" x 8" (100 mm x 200 mm), but its actual size is 3.5" x 7.25" (97 mm x 195 mm) after it



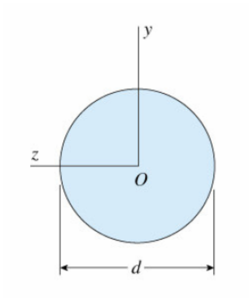
(a)

is surfaced

consider a rectangular of width b and depth h

$$S = \frac{b\,h^2}{6} = \frac{A\,h}{6} = 0.167\,A\,h$$

a rectangular cross section becomes more efficient as h increased, but very narrow section may fail because of lateral bucking



(b)

for a circular cross section of diameter d

$$S = \frac{\pi\,d^3}{32} = \frac{A\,d}{8} = 0.125\,A\,d$$

comparing the circular section to a square section of same area

$$h^2 = \pi\,d^2 / 4 \Rightarrow h = \sqrt{\pi}\,d / 2$$

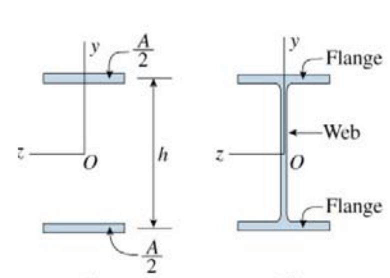
$$\frac{S_{square}}{S_{circle}} = \frac{0.167\,A\,h}{0.125\,A\,d} = \frac{0.167\sqrt{\pi}\,d/2}{0.125\,d} = \frac{0.148}{0.125} = 1.184$$

$\therefore$ the square section is more efficient than circular section

the most favorable case for a given area A and depth h would have to distribute $A / 2$ at a distance $h / 2$ from the neutral axis, then

$$I = \frac{A}{2}\left(\frac{h}{2}\right)^2 \times 2 = \frac{A\,h^2}{4}$$

$$S = \frac{I}{h/2} = \frac{A\,h}{2} = 0.5\,A\,h$$



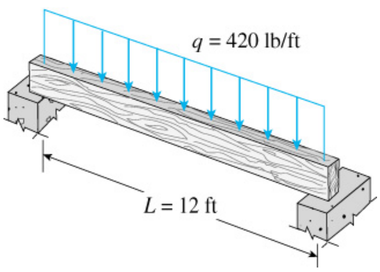
(c) (d)

the wide-flange section or an I - section with most material in the flanges would be the most efficient section for standard wide-flange beams, S is approximately

$$S \simeq 0.35\, A\, h$$

wide-flange section is more efficient than rectangular section of the same area and depth, $\because$ much of the material in rectangular beam is located near the neutral axis where it is unstressed, wide-flange section have most of the material located in the flanges, in addition, wide-flange shape is wider and therefore more stable with respect to sideways bucking

Example 5-5



a simply supported wood beam carries uniform load

$L = 3$ m $\qquad q = 4$ kN/m

$\sigma_{allow} = 12$ MPa $\qquad$ wood weights 5.4 kN/m^3

select the size of the beam

(a) calculate the required S

$$M_{max} = \frac{q\,L^2}{8} = \frac{(4 \text{ kN/m})\,(3 \text{ m})^2}{8} = 4.5 \text{ kN-m}$$

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{4.5 \text{ kN-m}}{12 \text{ MPa}} = 0.375 \times 10^6 \text{ mm}^3$$

(b) select a trial size for the beam (with lightest weight)

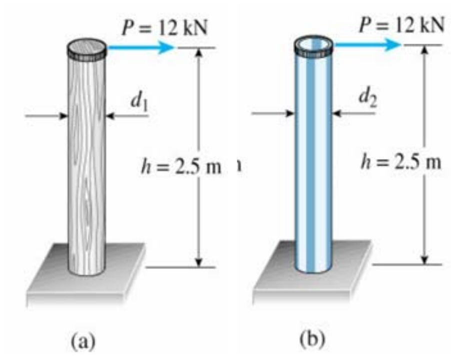
choose 75 x 200 beam, $S = 0.456 \times 10^6$ mm^3 and weight 77.11 N/m

(c) now the uniform load on the beam is increased to 77.11 N/m

$$S_{required} = (0.375 \times 10^6\ mm^3)\ \frac{4.077}{4.0} = 0.382 \times 10^6\ mm^3$$

(d) $S_{required} < S$ of 75 x 200 beam ($0.456 \times 10^6\ mm^3$) (O.K.)

Example 5-6



a vertical post 2.5 m high support a lateral load $P = 12$ kN at its upper end

(a) σ_{allow} for wood $= 15$ MPa

determine the diameter d_1

(b) σ_{allow} for aluminum tube $= 50$ MPa

determine the outer diameter d_2 if $t = d_2 / 8$

$$M_{max} = P\,h = 12 \times 2.5 = 30 \text{ kN-m}$$

(a) wood post

$$S_1 = \frac{\pi\, d_1^3}{32} = \frac{M_{max}}{\sigma_{allow}} = \frac{30 \times 10^3 \times 10^3}{15} = 2 \times 10^6\ mm^3$$

$$d_1 = 273 \text{ mm}$$

(b) aluminum tube

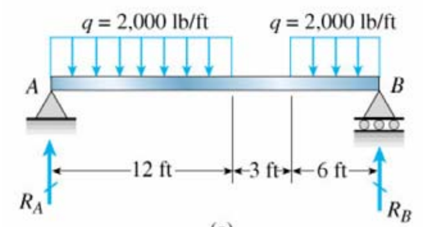
$$I_2 = \frac{\pi}{64}[d_2^4 - (d_2 - 2\,t)^4] = 0.03356\, d_2^4$$

$$S_2 = \frac{I_2}{c} = \frac{0.03356\, d_2^4}{d_2 / 2} = 0.06712\, d_2^3$$

$$S_2 = \frac{M_{max}}{\sigma_{allow}} = \frac{30 \times 10^3 \times 10^3}{50} = 600 \times 10^3\ mm^3$$

solve for d_2 => $d_2 = 208$ mm

Example 5-7



(a)

a simple beam AB of length 7 m

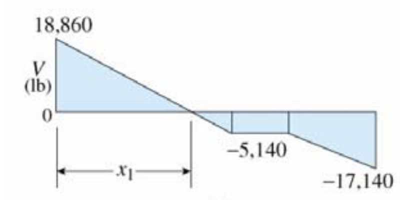
q = 60 kN/m σ_{allow} = 110 MPa

select a wide-flange shape

firstly, determine the support reactions

R_A = 188.6 kN R_B = 171.4 kN

the shear force V for $0 \leqq x \leqq 4$ m is



(b)

$$V = R_A - q\,x$$

for $V = 0$, the distance x_1 is

$$x_1 = \frac{R_A}{q} = \frac{188.6 \text{ kN}}{60 \text{ kN/m}} = 3.143 \text{ m}$$

and the maximum moment at the section is

$$M_{max} = 188.6 \text{ x } 3.143 / 2 = 296.3 \text{ kN-m}$$

the required section moudlus is

$$S = \frac{M_{max}}{\sigma_{allow}} = \frac{296.3 \text{ x } 10^6 \text{ N-mm}}{110 \text{ MPa}} = 2.694 \text{ x } 10^6 \text{ mm}^3$$

from table E-1, select the HE 450 A section with S = 2,896 cm^3

the weight of the beam is 140 kg/m, now recalculate the reactions, M_{max}, and $S_{required}$, we have

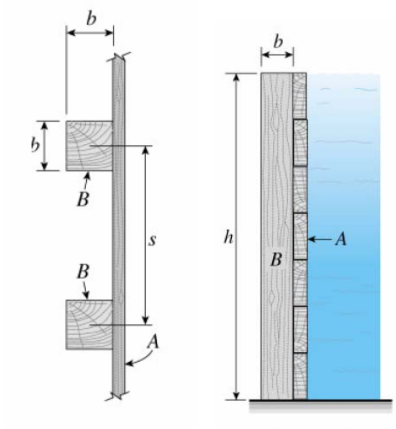
R_A = 193.4 kN R_B = 176.2 kN

V = 0 at x_1 = 3.151 m

=> M_{max} = 304.7 KN-m

$$S_{required} = \frac{M_{max}}{\sigma_{allow}} = 2{,}770 \text{ cm}^3 < 2{,}896 \text{ cm}^3 \qquad \text{(O. K.)}$$

Example 5-8



(a) Top view (b) Side view

the vertical posts B are supported planks A of the dam

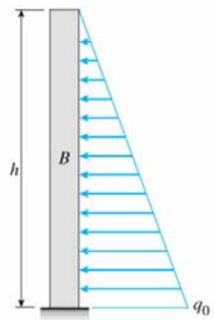
post B are of square section b x b

the spacing of the posts $s = 0.8$ m

water level $h = 2.0$ m

$\sigma_{allow} = 8.0$ MPa

determine b



(c) Loading diagram

the post B is subjected to the water pressure (triangularly distributed)

the maximum intensity q_0 is

$$q_0 = \gamma \, h \, s$$

the maximum bending moment occurs at the base is

$$M_{max} = \frac{q_0 \, h}{2} \left(\frac{h}{3}\right) = \frac{\gamma \, h^3 \, s}{6}$$

$$\text{and} \quad S = \frac{M_{max}}{\sigma_{allow}} = \frac{\gamma \, h^3 \, s}{6 \, \sigma_{allow}} = \frac{b^3}{6}$$

$$b^3 = \frac{\gamma \, h^3 \, s}{\sigma_{allow}} = \frac{9.81 \text{ x } 2^3 \text{ x } 0.8}{8 \text{ x } 10^6} = 0.007848 \text{ m}^3 = 7.848 \text{ x } 10^6 \text{ mm}^3$$

$b = 199$ mm use $b = 200$ mm

5.7 Nonprismatic Beams

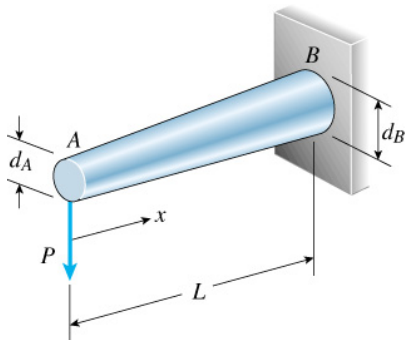
nonprismatic beams are commonly used to reduce weight and improve appearance, such beams are found in automobiles, airplanes, machinery, bridges, building etc.

$\sigma = M / S$, S varying with x, so we cannot assume that the maximum stress occur at the section with M_{max}

Example 5-9

a tapered cantilever beam AB of solid circular cross section supports a load P at the free end with $d_B / d_A = 2$

determine σ_B and σ_{max}



$$d_x = d_A + (d_B - d_A) \frac{x}{L}$$

$$S_x = \frac{\pi d_x^3}{32} = \frac{\pi}{32} [d_A + (d_B - d_A)\frac{x}{L}]^3$$

$\because$ $M_x = P x$, then the maximum bending stress at any cross section is

$$\sigma_1 = \frac{M_x}{S_x} = \frac{32 P x}{\pi [d_A + (d_B - d_A) (x/L)]^3}$$

at support B, $d_B = 2 d_A$, $x = L$, then

$$\sigma_B = \frac{4 P L}{\pi d_A^3}$$

to find the maximum stress in the beam, take $d\sigma_1 / dx = 0$

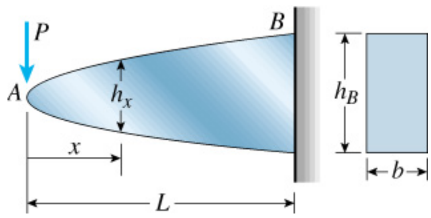
$$\Rightarrow x = L / 2$$

at that section ($x = L/2$), the maximum is

$$\sigma_{max} = \frac{128\, P\, L}{27\, \pi\, d_A^3} = 4.741\, \frac{P\, L}{\pi\, d_A^3}$$

it is 19% greater than the stress at the built-in end

Example 5-10



a cantilever beam of length L support a load P at the free end

cross section is rectangular with constant width b, the height may vary such that $\sigma_{max} = \sigma_{allow}$ for every cross section (fully stressed beam)

determine the height of the beam

$$M = P\, x \qquad S = \frac{b\, h_x^2}{6}$$

$$\sigma_{allow} = \frac{M}{S} = \frac{P\, x}{b\, h_x^2 / 6} = \frac{6\, P\, x}{b\, h_x^2}$$

solving the height for the beam, we have

$$h_x = \left(\frac{6\, P\, x}{b\, \sigma_{allow}}\right)^{1/2}$$

at the fixed end $(x = L)$

$$h_B = \left(\frac{6\, P\, L}{b\, \sigma_{allow}}\right)^{1/2}$$

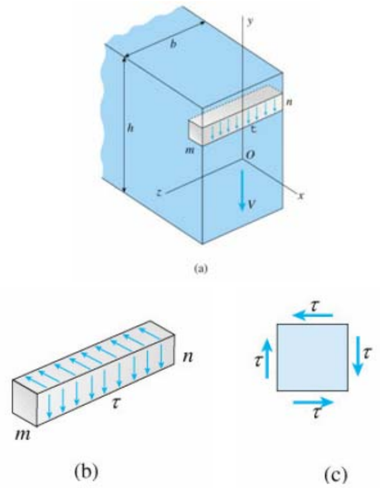
then $\qquad h_x = h_B \left(\dfrac{x}{L}\right)^{1/2}$ $\qquad$ the idealized beam has the parabolic shape

5-8 Shear Stress in Beam of Rectangular Cross Section

for a beam subjected to M and V with rectangular cross section having width b and height h, the shear stress τ acts parallel to the shear force V

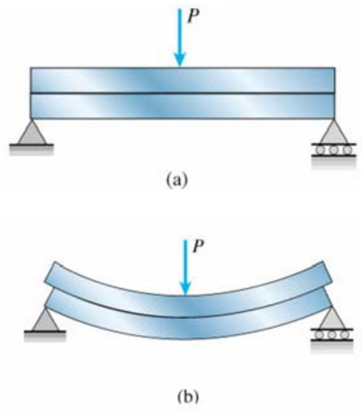
assume that τ is uniform across the width of the beam

consider a beam section subjected the a shear force V, we isolate a small element mn, the shear stresses τ act vertically and accompanied horizontally as shown



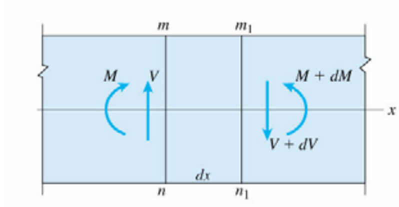
∵ the top and bottom surfaces are free, then the shear stress must be vanish, i.e. $\tau = 0$ at $y = \pm h/2$

for two equal rectangular beams of height h subjected to a concentrated load P, if no friction between the beams, each beam will be in compression above its N.A., the lower longitudinal line of the upper beam will slide w.r.t. the upper line of the lower beam



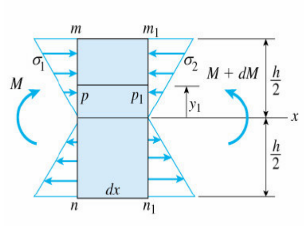
for a solid beam of height $2h$, shear stress must exist along N.A. to prevent sliding, thus single beam of depth $2h$ will much stiffer and stronger than two separate beams each of depth h

consider a small section of the beam subjected M and V in left face and $M + dM$ and $V + dV$ in right face



Side view of beam
(a)

for the element mm_1p_1p, τ acts on p_1p and no stress on mm_1

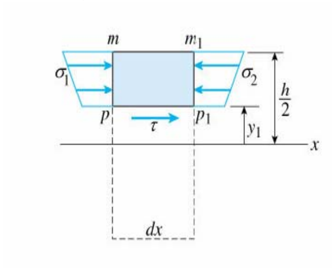


Side view of element
(b)

if the beam is subjected to pure bending (M = constant), σ_x acting on mp and m_1p_1 must be equal, then $\tau = 0$ on pp_1

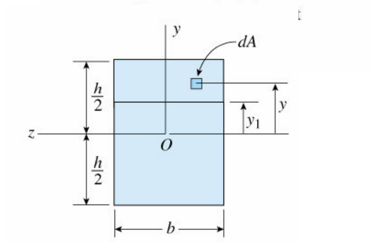
for nonuniform bending, M acts on mn and $M + dM$ acts on m_1n_1, consider dA at the distance y form N.A., then on mn

$$\sigma_x \, dA = \frac{M\,y}{I} dA$$



hence the total horizontal force on mp is

$$F_1 = \int \frac{M\,y}{I} dA$$



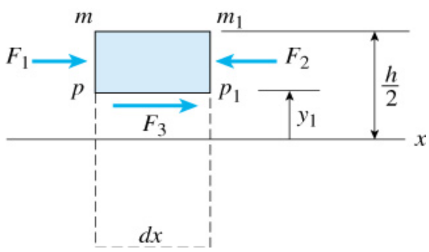
Cross section of beam at subelement
(d)

similarly

$$F_2 = \int \frac{(M + dM)\,y}{I} dA$$

and the horizontal force on pp_1 is

$$F_3 = \tau \, b \, dx$$



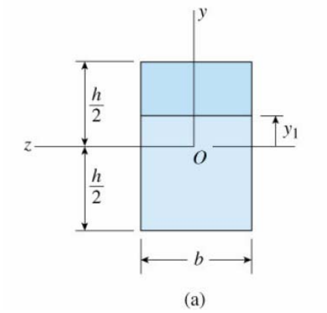
equation of equilibrium

$$F_3 = F_2 - F_1$$

$$\tau\, b\, dx \;=\; \int \frac{(M + dM)\, y}{I}\, dA \;-\; \int \frac{M\, y}{I}\, dA$$

$$\tau \;=\; \frac{dM}{dx}\,\frac{1}{Ib} \int y\, dA \;=\; \frac{V}{I\, b} \int y\, dA$$

denote $Q = \int y\, dA$ is the first moment of the cross section area above the level y (area mm_1p_1p) at which the shear stress τ acts, then

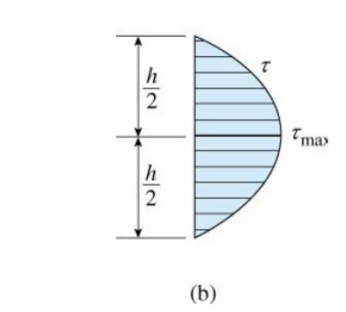


$$\tau \;=\; \frac{V\, Q}{I\, b} \qquad \text{shear stress formula}$$

for V, I, b are constants, $\tau \sim Q$

for a rectangular cross section

$$Q \;=\; b\,(\frac{h}{2} - y_1)\,(y_1 + \frac{h/2 - y_1}{2}) \;=\; \frac{b}{2}\,(\frac{h^2}{4} - y_1^{\,2})$$



then $\qquad \tau \;=\; \dfrac{V}{2\, I}\,(\dfrac{h^2}{4} - y_1^{\,2})$

$\tau = 0$ at $y_1 = \pm\, h/2$, τ_{max} occurs at $y_1 = 0$ (N.A.)

$$\tau_{max} \;=\; \frac{V\, h^2}{8\, I} \;=\; \frac{V\, h^2}{8\, b\, h^3/12} \;=\; \frac{3\, V}{2\, A} \;=\; \frac{3}{2}\, \tau_{ave}$$

τ_{max} is 50% larger than τ_{ave}

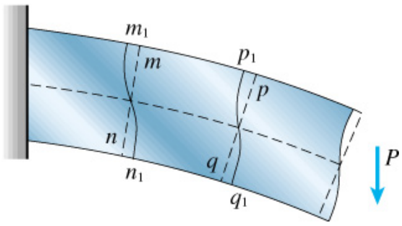
$\because$ V = resultant of shear stress, $\therefore$ V and τ in the same direction

Limitations

the shear formula are valid only for beams of linear elastic material with small deflection

the shear formula may be consider to be exact for narrow beam ($\because$ τ is assumed constant across b), when $b = h$, true τ_{max} is about 13% larger than the value given by the shear formula

Effects of Shear Strains



$\because$ τ vary parabolically from top to bottom, and $\gamma = \tau / G$ must vary in the same manner

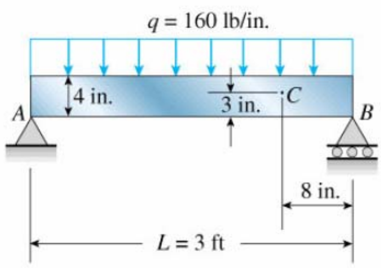
thus the cross sections were plane surfaces become warped, no shear strains occur on the surfaces, and maximum shear strain occurs on N.A.

$\because$ $\gamma_{max} = \tau_{max} / G$, if V remains constant along the beam, the warping of all sections is the same, i.e. $mm_1 = pp_1 = \cdots$, the stretching or shortening of the longitudinal lines produced by the bending moment is unaffected by the shear strain, and the distribution of the normal stress σ is the same as it is in pure bending

for shear force varies continuously along the beam, the warping of cross sections due to shear strains does not substantially affect the longitudinal strains by more experimental investigation

thus, it is quite justifiable to use the flexure formula in the case of nonuniform bending, except the region near the concentrate load acts of irregularly change of the cross section (stress concentration)

Example 5-11

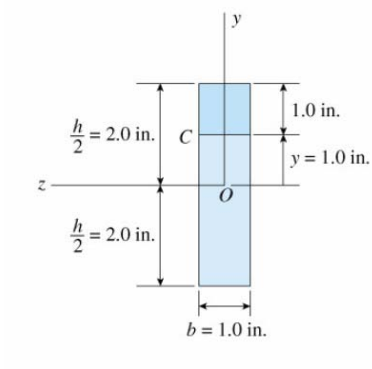


(a)

a metal beam with span $L = 1$ m

$q = 28$ kN/m $b = 25$ mm $h = 100$ mm

determine σ_C and τ_C at point C



(b)

the shear force V_C and bending moment M_C at the section through C are found

$$M_C = 2.24 \text{ kN-m}$$
$$V_C = -8.4 \text{ kN}$$

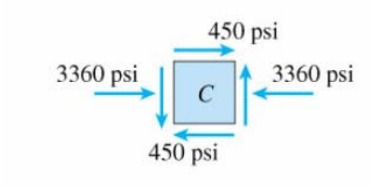
the moment of inertia of the section is

$$I = \frac{b\,h^3}{12} = \frac{1}{12} \text{ x } 25 \text{ x } 100^3 = 2{,}083 \text{ x } 10^3 \text{ mm}^4$$

normal stress at C is

$$\sigma_C = -\frac{M\,y}{I} = -\frac{2.24 \text{ x } 106 \text{ N-mm x } 25 \text{ mm}}{2{,}083 \text{ x } 10^3 \text{ mm}^4} = -26.9 \text{ MPa}$$

shear stress at C, calculate Q_C first



(c)

$A_C = 25 \text{ x } 25 = 625 \text{ mm}^2$ $\quad y_C = 37.5$ mm

$$Q_C = A_C\,y_C = 23{,}400 \text{ mm}^3$$

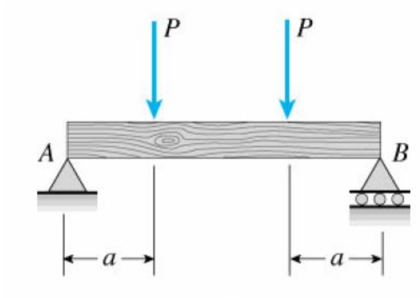
$$\tau_C = \frac{V_C\,Q_C}{I\,b} = \frac{8{,}400 \text{ x } 23{,}400}{2{,}083 \text{ x } 10^3 \text{ x } 25} = 3.8 \text{ MPa}$$

the stress element at point C is shown

Example 5-12

a wood beam AB supporting two concentrated loads P

$b = 100$ mm $\quad h = 150$ mm



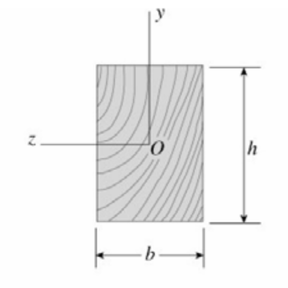
(a)

$a \quad = \quad 0.5 \text{ m} \qquad \sigma_{allow} \quad = \quad 11 \text{ MPa} \qquad \tau_{allow} \quad = \quad 1.2 \text{ MPa}$

determine P_{max}

the maximum shear force and bending moment are

$$V_{max} = P \qquad M_{max} = P\, a$$



(b)

the section modulus and area are

$$S = \frac{b\, h^2}{6} \qquad A = b\, h$$

maximum normal and shear stresses are

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6\, P\, a}{b\, h^2} \qquad \tau_{max} = \frac{3\, V_{max}}{2\, A} = \frac{3\, P}{2\, b\, h}$$

$$P_{bending} = \frac{\sigma_{allow}\, b\, h^2}{6\, a} = \frac{11 \text{ x } 100 \text{ x } 150^2}{6 \text{ x } 500} = 8{,}250 \text{ N} = 8.25 \text{ kN}$$

$$P_{shear} = \frac{2\, \tau_{allow}\, b\, h}{3} = \frac{2 \text{ x } 1.2 \text{ x } 100 \text{ x } 150}{3} = 12{,}000 \text{ N} = 12 \text{ kN}$$

$$\therefore \quad P_{max} = 8.25 \text{ kN}$$

8-9 Shear Stresses in Beam of Circular Cross Section

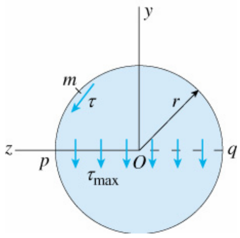
$$\tau = \frac{V\, Q}{I\, b} \qquad I = \frac{\pi\, r^4}{4} \quad \text{for solid section}$$

the shear stress at the neutral axis

$$Q = A\, y = \left(\frac{\pi\, r^2}{2}\right)\left(\frac{4\, r}{3\, \pi}\right) = \frac{2\, r^3}{3} \qquad b = 2\, r$$

$$\tau_{max} = \frac{V(2r^3/3)}{(\pi r^4/4)(2r)} = \frac{4V}{3\pi r^2} = \frac{4V}{3A} = \frac{4}{3}\tau_{ave}$$

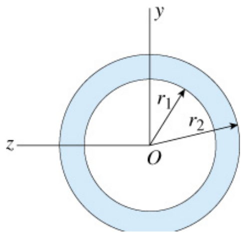
for a hollow circular cross section



$$I = \frac{\pi}{4}(r_2^4 - r_1^4) \qquad Q = \frac{2}{3}(r_2^3 - r_1^3)$$

$$b = 2(r_2 - r_1)$$

then the maximum shear stress at N.A. is



$$\tau_{max} = \frac{VQ}{Ib} = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right)$$

where $\qquad A = \pi(r_2^2 - r_1^2)$

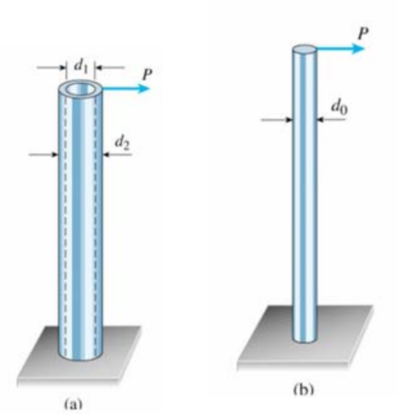
Example 5-13

a vertical pole of a circular tube

$d_2 = 100$ mm $\quad d_1 = 80$ mm $\quad P = 6{,}675$ N

(a) determine the τ_{max} in the pole

(b) for same P and same τ_{max}, calculate d_0 of a solid circular pole



(a) The maximum shear stress of a circular tube is

$$\tau_{max} = \frac{4P}{3\pi}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^4 - r_1^4}\right)$$

for $\quad P = 6{,}675$ N $\qquad r_2 = 50$ mm $\qquad r_1 = 40$ mm

$$\tau_{max} = 4.68 \text{ MPa}$$

(b) for a solid circular pole, τ_{max} is

$$\tau_{max} = \frac{4P}{3\pi (d_0/2)^2}$$

$$d_0^2 = \frac{16P}{3\pi\tau_{max}} = \frac{16 \times 6{,}675}{3\pi \times 4.68} = 2.42 \times 10^{-3} \text{ m}^2$$

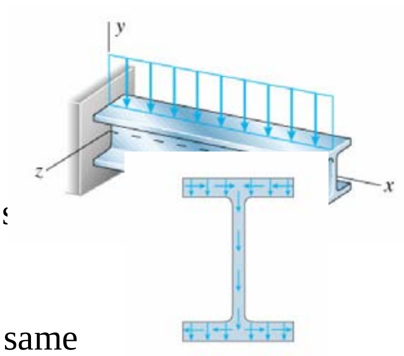
then $d_0 = 49.21$ mm

the solid circular pole has a diameter approximately 5/8 that of the tubular pole

5-10 Shear Stress in the Webs of Beams with Flanges

for a beam of wide-flange shape subjected to shear force V, shear stress is much more complicated

most of the shear force is carried by stresses in the web

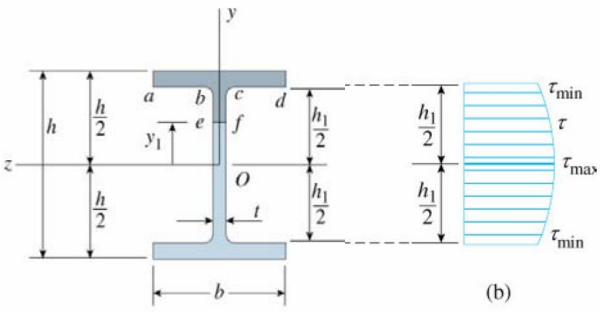


(b)

consider the shear stress at ef, the same assumption as in the case in rectangular beam, i.e. τ // y axis and uniformly distributed across t

$$\tau = \frac{VQ}{Ib} \quad \text{is still valid with} \quad b = t$$

the first moment Q of the shaded area is divided into two parts, i.e. the upper flange and the area between bc and ef in the web



(a) (b)

$$A_1 = b\left(\frac{h}{2} - \frac{h_1}{2}\right) \qquad A_2 = t\left(\frac{h_1}{2} - y_1\right)$$

then the first moment of A_1 and A_2 w.r.t. N.A. is

$$Q = A_1\left(\frac{h_1}{2} + \frac{h/2 - h_1/2}{2}\right) + A_2\left(y_1 + \frac{h_1/2 - y_1}{2}\right)$$

$$= \frac{b}{8}(h^2 - h_1^2) + \frac{t}{8}(h_1^2 - 4\,y_1^2)$$

$$\tau = \frac{V\,Q}{I\,b} = \frac{V}{8\,I\,t}\left[\frac{b}{8}(h^2 - h_1^2) + \frac{t}{8}(h_1^2 - 4\,y_1^2)\right]$$

where $$I = \frac{b\,h^3}{12} - \frac{(b - t)\,h_1^3}{12} = \frac{1}{12}(b\,h^3 - b\,h_1^3 + t\,h_1^3)$$

maximum shear stress in the web occurs at N.A., $y_1 = 0$

$$\tau_{max} = \frac{V}{8\,I\,t}(b\,h^2 - b\,h_1^2 + t\,h_1^2)$$

minimum shear stress occurs where the web meets the flange, $y_1 = \pm h_1/2$

$$\tau_{min} = \frac{V\,b}{8\,I\,t}(h^2 - h_1^2)$$

the maximum stress in the web is from 10% to 60% greater than the minimum stress

the shear force carried by the web consists two parts, a rectangle of area $h_1\,\tau_{min}$ and a parabolic segment of area $\frac{2}{3}\,h_1\,(\tau_{max} - \tau_{min})$

$$V_{web} = h_1\,\tau_{min} + \frac{2}{3}\,h_1\,(\tau_{max} - \tau_{min})$$

$$= \frac{t\, h_1}{3} (2\, \tau_{max} + \tau_{min})$$

V_{web} = 90% ~ 98% of total V

for design work, the approximation to calculate $\tau_{\max}$ is

$$\tau_{max} = \frac{V}{t\, h_1} \quad \begin{matrix} \Leftarrow \text{ total shear force} \\ \Leftarrow \text{ web area} \end{matrix}$$

for typical wide-flange beam, error is within ± 10%

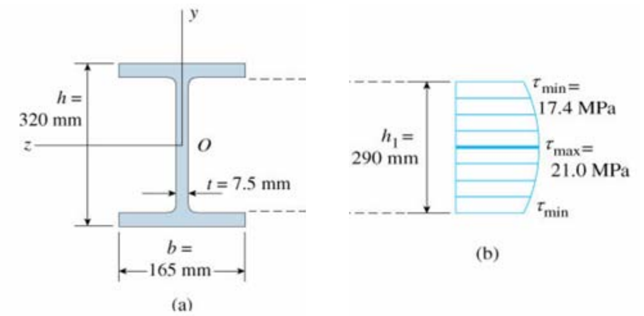
when considering y in the flange, constant τ across b cannot be made, e.g. at $y_1 = h_1/2$, t at ab and cd must be zero, but on bc, $\tau = \tau_{min}$

actually the stress is very complicated here, the stresses would become very large at the junction if the internal corners were square

Example 5-14

a beam of wide-flange shape with b = 165 mm, t = 7.5mm, h = 320 mm, and h_1 = 290 mm, vertical shear force V = 45 kN

determine τ_{max}, τ_{min} and total shear force in the web



the moment of inertia of the cross section is

$$I = \frac{1}{12} (b\, h^3 - b\, h_1^{\,3} + t\, h_1^{\,3}) = 130.45 \text{ x } 10^6 \text{ mm}^4$$

the maximum and minimum shear stresses are

$$\tau_{max} \quad = \quad \frac{V}{8\,I\,t}\,(b\,h^2 - b\,h_1^{\,2} + t\,h_1^{\,2}) \quad = \quad 21.0 \text{ MPa}$$

$$\tau_{min} \quad = \quad \frac{V\,b}{8\,I\,t}\,(h^2 - h_1^{\,2}) \quad = \quad 17.4 \text{ MPa}$$

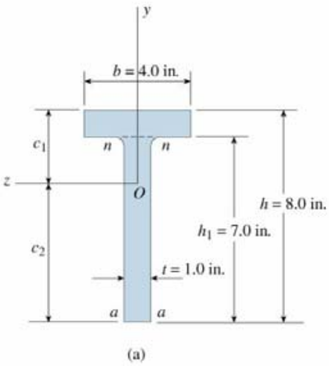
the total shear force is

$$V_{\text{web}} \quad = \quad \frac{t\,h_1}{3}\,(2\,\tau_{max} + \tau_{min}) \quad = \quad 43.0 \text{ kN}$$

tnd the average shear stress in the web is

$$\tau_{ave} \quad = \quad \frac{V}{t\,h_1} \quad = \quad 20.7 \text{ MPa}$$

Example 5-15



(a)

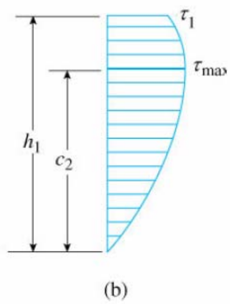
a beam having a T-shaped cross section

b = 100 mm t = 24 mm h = 200 mm

V = 45 kN

determine τ_{nn} (top of the web) and τ_{max}

$$c_1 \quad = \quad \frac{76 \text{ x } 24 \text{ x } 12 + 200 \text{ x } 24 \text{ x } 100}{76 \text{ x } 24 + 200 \text{ x } 24} \quad = \quad 75.77 \text{ mm}$$



(b)

$$c_2 \quad = \quad 200 \quad - \quad c_1 \quad = \quad 124.33 \text{ mm}$$

$$I \quad = \quad I_{aa} \quad - \quad A\,c_2^{\,2}$$

$$I_{aa} \quad = \quad \frac{b\,h^3}{3} \quad - \quad \frac{(b - t)\,h_1^{\,3}}{3} \quad = \quad 128.56 \text{ x } 10^6 \text{ mm}^3$$

$$A\,c_2^{\,2} \quad = \quad 102.23 \text{ x } 10^6 \text{ mm}^3$$

$$I \quad = \quad 26.33 \text{ x } 10^6 \text{ mm}^3$$

to find the shear stress τ_1 (τ_{nn}), calculate Q_1 first

$$Q_1 \quad = \quad 100 \text{ x } 24 \text{ x } (75.77 - 012) \quad = \quad 153 \text{ x } 10^3 \text{ mm}^3$$

$$\tau_1 \quad = \quad \frac{V\, Q_1}{I\, t} \quad = \quad \frac{45 \text{ x } 10^3 \text{ x } 153 \text{ x } 10^3}{26.33 \text{ x } 10^6 \text{ x } 24} \quad = \quad 10.9 \text{ MPa}$$

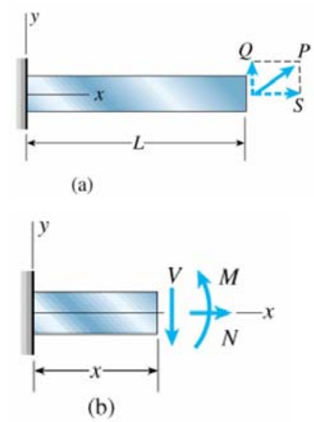
to find τ_{max}, we want to find Q_{max} at N.A.

$$Q_{max} \quad = \quad t\, c_2\, (c_2/2) \quad = 24 \text{ x } 124.23 \text{ x } (124.23/2) = 185 \text{ x } 10^3 \text{ mm}^3$$

$$\tau_{max} \quad = \quad \frac{V\, Q_{max}}{I\, t} \quad = \quad \frac{45 \text{ x } 10^3 \text{ x } 185 \text{ x } 10^3}{26.33 \text{ x } 10^6} \quad = \quad 13.2 \text{ MPa}$$

5.11 Built-up Beams and Shear Flow

5.12 Beams with Axial Loads



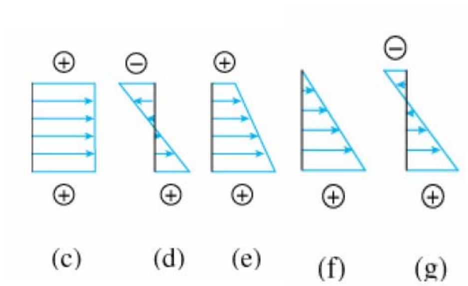
beams may be subjected to the simultaneous action of bending loads and axial forces,

e.g. cantilever beam subjected to an inclined force P, it may be resolved into two components Q and S, then

$$M = Q\,(L - x) \qquad V = -\,Q \qquad N = S$$

and the stresses in beams are

$$\sigma = -\,\frac{M\,y}{I} \qquad \tau = \frac{V\,Q}{I\,b} \qquad \sigma = \frac{N}{A}$$



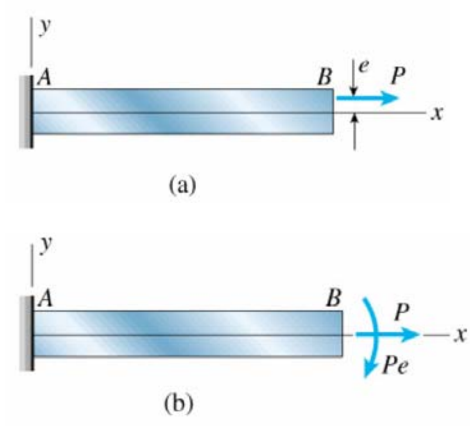
the final stress distribution can be obtained by combining the stresses associated with each stress resultant

$$\sigma = -\frac{M y}{I} + \frac{N}{A}$$

whenever bending and axial loads act simultaneously, the neutral axis no longer passes through the centroid of the cross section

Eccentric Axial Loads

a load P acting at distance e from the x axis, e is called eccentricity



$$N = P \qquad M = - P e$$

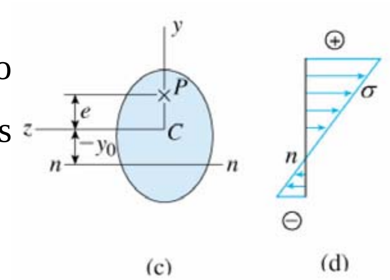
then the normal stress at any point is

$$\sigma = \frac{P e y}{I} + \frac{P}{A}$$

the position of the N.A. nn can be obtained by setting $\sigma = 0$

$$y_0 = -\frac{I}{A e}$$ minus sign shows the N.A. lies below z-axis

if e increased, N.A. moves closer to the centroid, if e reduced, N.A. moves away from the centroid

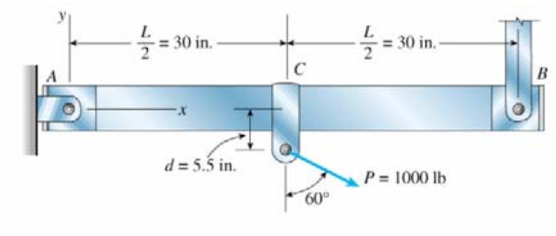


Example 5-15

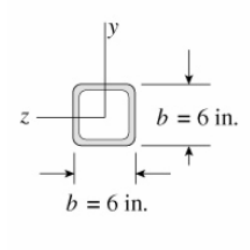
a tubular beam ACB of length $L = 1.5$ m loaded by a inclined force P at mid length

$P = 4.5$ kN $\qquad d = 140$ mm $\qquad b = 150$ mm $\qquad A = 12{,}500$ mm^2

$$I \quad = \quad 33.86 \text{ x } 10^6 \text{ mm}^4$$



(a)



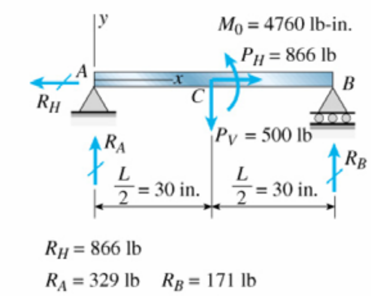
(b)

$$P_h \quad = \quad P \sin 60^o \quad = \quad 3{,}897 \text{ N}$$

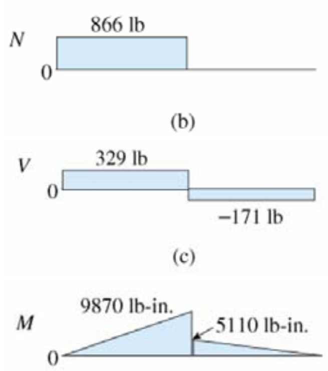
$$P_v \quad = \quad P \cos 60^o \quad = \quad 2{,}250 \text{ N}$$

$$M_0 \quad = \quad P_h\, d \quad = \quad 3{,}897 \text{ x } 140 \quad = \quad 545.6 \text{ x } 10^3 \text{ N-mm}$$

the axial force, shear force and bending moment diagrams are sketched first



(a)



(b)
(c)
(d)

the maximum tensile stress occurs at the bottom of the beam, y = - 75 mm

$$(\sigma_t)_{max} \quad = \quad \frac{N}{A} \; - \; \frac{M\,y}{I} \quad = \quad \frac{3{,}897}{12{,}500} \; - \; \frac{1{,}116.8 \text{ x } 10^3\,(-75)}{33.86 \text{ x } 10^6}$$

$$= \quad 0.312 \quad + \quad 2.474 \quad = \quad 2.79 \text{ MPa}$$

the maximum compressive stress occurs at the top of the beam, y = 75 mm

$$(\sigma_c)_{left} \quad = \quad \frac{N}{A} \; - \; \frac{M\,y}{I} \quad = \quad \frac{3{,}897}{12{,}500} \; - \; \frac{1{,}116.8 \text{ x } 10^3 \text{ x } 75}{33.86 \text{ x } 10^6}$$

$$= \quad 0.312 \quad - \quad 2.474 \quad = \quad -\,2.16 \text{ MPa}$$

$$(\sigma_c)_{right} \quad = \quad \frac{N}{A} \quad - \quad \frac{M\,y}{I} \quad = \quad 0 \quad - \quad \frac{571.2\ 10^3 \text{ x } 75}{33.86 \text{ x } 10^6}$$

$$= \quad -\,1.265 \text{ MPa}$$

thus $(\sigma_c)_{max} \quad = \quad -\,2.16$ MPa occurs at the top of the beam to the left of point C

5.13 Stress Concentration in Beams