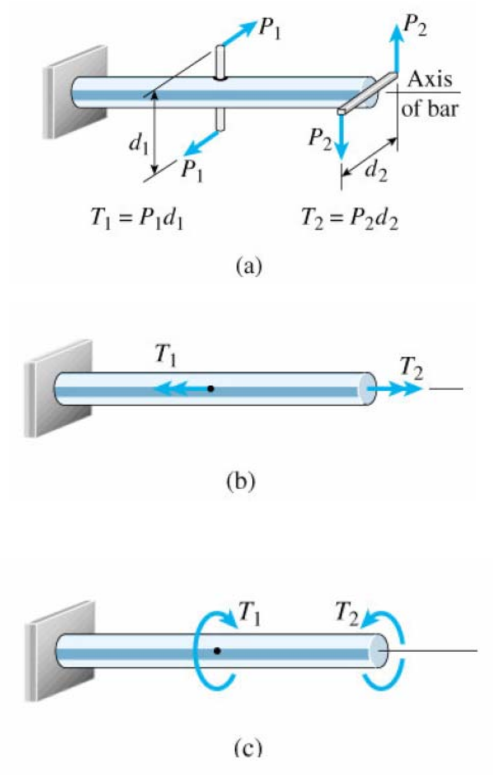
Chapter 3 Torsion

3.1 Introduction



Torsion : twisting of a structural member, when it is loaded by couples that produce rotation about its longitudinal axis

$$T_1 = P_1 d_1 \qquad T_2 = P_2 d_2$$

the couples T_1, T_2 are called torques, twisting couples or twisting moments

unit of T : N-m, lb-ft

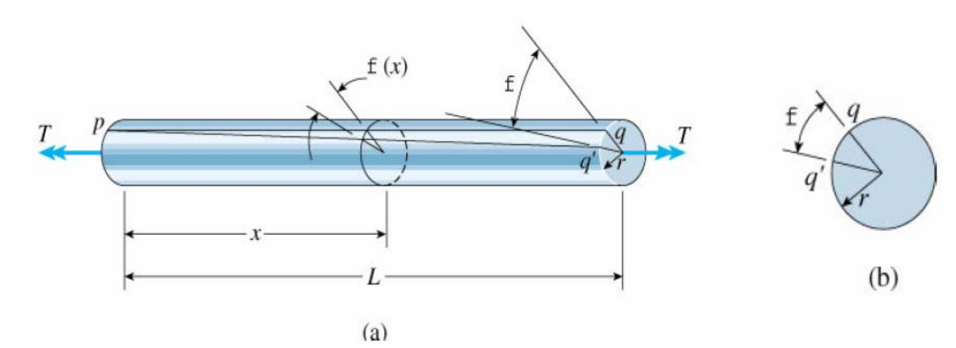
in this chapter, we will develop formulas for the stresses and deformations produced in circular bars subjected to torsion, such as drive shafts, thin-walled members

analysis of more complicated shapes required more advanced method then those presented here

this chapter cover several additional topics related to torsion, such statically indeterminate members, strain energy, thin-walled tube of noncircular section, stress concentration, and nonlinear behavior

3.2 Torsional Deformation of a Circular Bar

consider a bar or shaft of circular cross section twisted by a couple T, assume the left-hand end is fixed and the right-hand end will rotate a small angle ϕ, called angle of twist

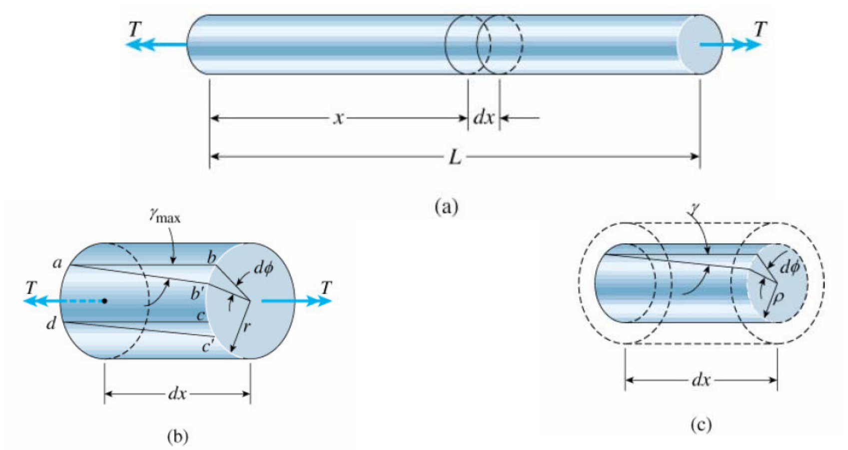


if every cross section has the same radius and subjected to the same torque, the angle $\phi(x)$ will vary linearly between ends

under twisting deformation, it is assumed

1. plane section remains plane
2. radii remaining straight and the cross sections remaining plane and circular
3. if ϕ is small, neither the length L nor its radius will change

consider an element of the bar dx, on its outer surface we choose an small element $abcd$,



during twisting the element rotate a small angle $d\phi$, the element is in a state of pure shear, and deformed into $ab'c'd$, its shear strain γ_{max} is

$$\gamma_{max} = \frac{b\,b'}{a\,b} = \frac{r\,d\phi}{dx}$$

$d\phi / dx$ represents the rate of change of the angle of twist ϕ, denote $\theta = d\phi / dx$ as the angle of twist per unit length or the rate of twist, then

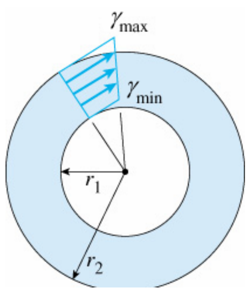
$$\gamma_{max} = r\,\theta$$

in general, ϕ and θ are function of x, in the special case of pure torsion, θ is constant along the length (every cross section is subjected to the same torque)

$$\theta = \frac{\phi}{L} \qquad \text{then} \qquad \gamma_{max} = \frac{r\,\phi}{L}$$

and the shear strain inside the bar can be obtained

$$\gamma = \rho\,\theta = \frac{\rho}{r}\gamma_{max}$$



for a circular tube, it can be obtained

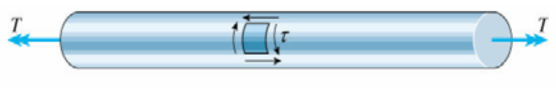
$$\gamma_{min} = \frac{r_1}{r_2}\gamma_{max}$$

the above relationships are based only upon geometric concepts, they are valid for a circular bar of any material, elastic or inelastic, linear or nonlinear

3.3 Circular Bars of Linearly Elastic Materials

shear stress τ in the bar of a linear elastic material is

$$\tau = G\,\gamma$$



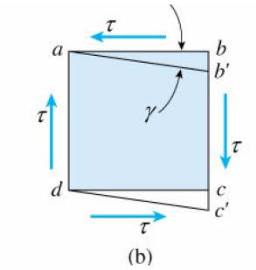
(a)

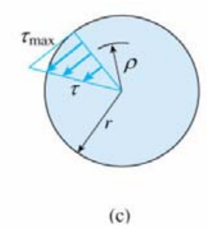
G : shear modulus of elasticity

with the geometric relation of the shear strain, it is obtained

$$\tau_{max} = G\, r\, \theta$$

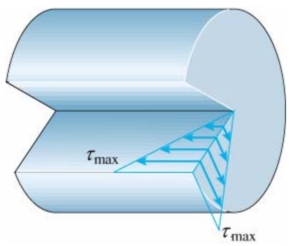
$$\tau = G\, \rho\, \theta = \frac{\rho}{r}\, \tau_{max}$$





τ and γ in circular bar vary linear with the radial distance ρ from the center, the maximum values τ_{max} and γ_{max} occur at the outer surface

the shear stress acting on the plane of the cross section are accompanied by shear stresses of the same magnitude acting on longitudinal plane of the bar

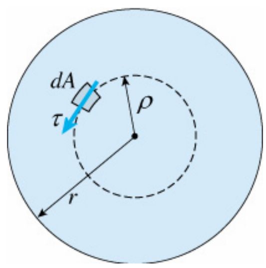


if the material is weaker in shear on longitudinal plane than on cross-sectional planes, as in the case of a circular bar made of wood, the first crack due to twisting will appear on the surface in longitudinal direction

a rectangular element with sides at 45^{o} to the axis of the shaft will be subjected to tensile and compressive stresses

The Torsion Formula

consider a bar subjected to pure torsion, the shear force acting on an element dA is $\tau\, dA$, the moment of this force about the axis of bar is $\tau\, \rho\, dA$



$$dM = \tau\, \rho\, dA$$

equation of moment equilibrium

$$T = \int_A dM = \int_A \tau \rho \, dA = \int_A G \theta \rho^2 \, dA = G \theta \int_A \rho^2 \, dA$$

$$= G \theta I_p \qquad [\tau = G \theta \rho]$$

in which $I_p = \int_A \rho^2 \, dA$ is the polar moment of inertia

$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad \text{for circular cross section}$$

the above relation can be written

$$\theta = \frac{T}{G I_p}$$

$G I_p$: torsional rigidity

the angle of twist ϕ can be expressed as

$$\phi = \theta L = \frac{T L}{G I_p} \qquad \phi \text{ is measured in radians}$$

torsional flexibility $f = \dfrac{L}{G I_p}$

torsional stiffness $k = \dfrac{G I_p}{L}$

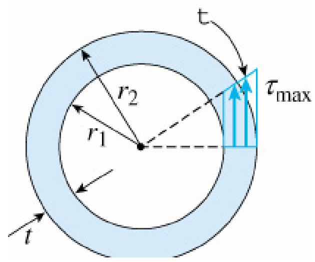
and the shear stress is

$$\tau = G \rho \theta = G \rho \frac{T}{G I_p} = \frac{T \rho}{I_p}$$

the maximum shear stress $\tau_{\max}$ at $\rho = r$ is

$$\tau_{max} = \frac{T\,r}{I_p} = \frac{16\,T}{\pi\,d^3}$$

for a circular tube



$$I_p = \pi\,(r_2^4 - r_1^4) / 2 = \pi\,(d_2^4 - d_1^4) / 32$$

if the hollow tube is very thin

$$I_p \simeq \pi\,(r_2^2 + r_1^2)\,(r_2 + r_1)\,(r_2 - r_1) / 2$$

$$= \pi\,(2r^2)\,(2r)\,(t) = 2\,\pi\,r^3\,t = \pi\,d^3\,t / 4$$

limitations

1. bar have circular cross section (either solid or hollow)
2. material is linear elastic

note that the above equations cannot be used for bars of noncircular shapes, because their cross sections do not remain plane and their maximum stresses are not located at the farthest distances from the midpoint

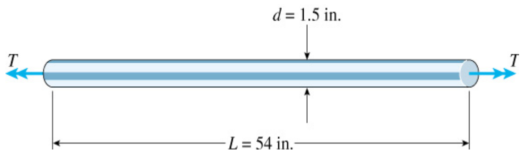
Example 3-1

a solid bar of circular cross section

d = 40 mm, L = 1.3 m, G = 80 GPa

(a) T = 340 N-m, τ_{max}, ϕ = ?

(b) τ_{all} = 42 MPa, $\phi_{all} = 2.5^o$, T = ?



(a) $$\tau_{max} = \frac{16\,T}{\pi\,d^3} = \frac{16 \times 340 \text{ N-M}}{\pi\,(0.04 \text{ m})^3} = 27.1 \text{ MPa}$$

$$I_p = \pi\,d^4 / 32 = 2.51 \times 10^{-7} \text{ m}^4$$

$$\phi = \frac{T\,L}{G\,I_p} = \frac{340 \text{ N-m} \times 1.3 \text{ m}}{80 \text{ GPa} \times 2.51 \times 10^{-7} \text{ m}^4} = 0.02198 \text{ rad} = 1.26^o$$

(b) due to τ_{all} = 42 MPa

$T_1 = \pi d^3 \tau_{all} / 16 = \pi (0.04 \text{ m})^3 \times 42 \text{ MPa} / 16 = 528$ N-m

due to $\phi_{all} = 2.5^o = 2.5 \times \pi \text{ rad} / 180^o = 0.04363$ rad

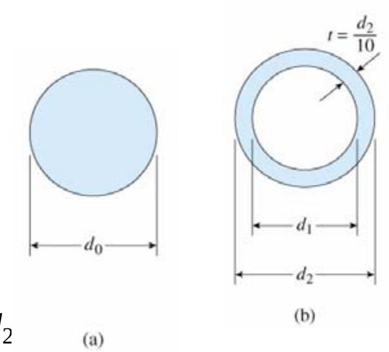
$T_2 = G I_p \phi_{all} / L = 80 \text{ GPa} \times 2.51 \times 10^{-7} \text{ m}^4 \times 0.04363 / 1.3 \text{ m}$

$= 674$ N-m

thus $T_{all} = \min [T_1, T_2] = 528$ N-m

Example 3-2

a steel shaft of either solid bar or circular tube



$T = 1200$ N-m, $\tau_{all} = 40$ MPa

$\theta_{all} = 0.75^o$ / m $G = 78$ GPa

(a) determine d_0 of the solid bar

(b) for the hollow shaft, $t = d_2 / 10$, determine d_2

(c) determine d_2 / d_0, $W_{\text{hollow}} / W_{\text{solid}}$

(a) for the solid shaft, due to τ_{all} = 40 MPa

$d_0^3 = 16 T / \pi \tau_{all} = 16 \times 1200 / \pi 40 = 152.8 \times 10^{-6} \text{ m}^3$

$d_0 = 0.0535 \text{ m} = 53.5$ mm

due to $\theta_{all} = 0.75^o / \text{m} = 0.75 \times \pi \text{ rad} / 180^o / \text{m} = 0.01309$ rad / m

$I_p = T / G \theta_{all} = 1200 / 78 \times 10^9 \times 0.01309 = 117.5 \times 10^{-8} \text{ m}^4$

$d_0^4 = 32 I_p / \pi = 32 \times 117.5 \times 10^{-8} / \pi = 1197 \times 10^{-8} \text{ m}^4$

$d_0 = 0.0588 \text{ m} = 58.8$ mm

thus, we choose $d_0 = 58.8$ mm [in practical design, $d_0 = 60$ mm]

(b) for the hollow shaft

$d_1 = d_2 - 2t = d_2 - 0.2 d_2 = 0.8 d_2$

$I_p \quad = \pi\,(d_2^4 - d_1^4) / 32 \quad = \pi\,[d_2^4 - (0.8d_2)^4] / 32 \quad = 0.05796\, d_2^4$

due to $\tau_{all} = 40$ MPa

$I_p = 0.05796\, d_2^4 = T\,r / \tau_{all} = 1200\,(d_2/2) / 40$

$d_2^3 = 258.8 \times 10^{-6}$ m^3

$d_2 = 0.0637$ m $= 63.7$ mm

due to $\theta_{all} = 0.75^o$ / m $= 0.01309$ rad / m

$\theta_{all} = 0.01309 = T / G\,I_p = 1200 / 78 \times 10^9 \times 0.05796\, d_2^4$

$d_2^4 = 2028 \times 10^{-8}$ m^4

$d_2 = 0.0671$ m $= 67.1$ mm

thus, we choose $d_0 = 67.1$ mm [in practical design, $d_0 = 70$ mm]

(c) the ratios of hollow and solid bar are

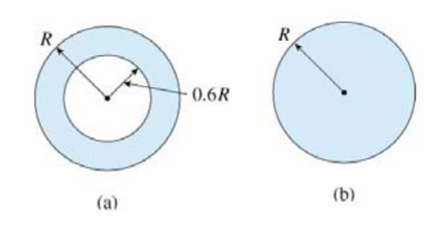
$d_2 / d_0 = 67.1 / 58.8 = 1.14$

$$\frac{W_{\text{hollow}}}{W_{\text{solid}}} = \frac{A_{\text{hollow}}}{A_{\text{solid}}} = \frac{\pi\,(d_2^2 - d_1^2)/4}{\pi\, d_0^2/4} = 0.47$$

the hollow shaft has 14% greater in diameter but 53% less in weight

Example 3-3

a hollow shaft and a solid shaft has same material, same length, same outer radius R, and $r_i = 0.6\,R$ for the hollow shaft



(a) for same T, compare their τ, θ, and W

(b) determine the strength-to-weight ratio

(a) $\because\ \tau = T\,R / I_p \qquad \theta = T\,L / G\,I_p$

$\therefore$ the ratio of τ or θ is the ratio of $1 / I_p$

$(I_p)_H = \pi\, R^2 /2 - \pi\,(0.6R)^2 /2 = 0.4352\, \pi\, R^2$

$(I_p)_S \quad = \quad \pi R^2 / 2 \quad = \quad 0.5\,\pi R^2$

$(I_p)_S / (I_p)_H \quad = \quad 0.5 / 0.4352 \quad = \quad 1.15$

thus $\quad \beta_1 \quad = \quad \tau_H / \tau_S \quad = \quad (I_p)_S / (I_p)_H \quad = \quad 1.15$

also $\quad \beta_2 \quad = \quad \phi_H / \phi_S \quad = \quad (I_p)_S / (I_p)_H \quad = \quad 1.15$

$\beta_3 \quad = W_H / W_S = A_H / A_S = \pi\,[R^2 - (0.6R)^2] / \pi R^2 \quad = 0.64$

the hollow shaft has 15% greater in τ and ϕ, but 36% decrease in weight

(b) strength-to-weight ratio $\quad S \quad = \quad T_{all} / W$

$T_H \quad = \tau_{\max} I_p / R \quad = \tau_{\max} (0.4352\,\pi R^4) / R \quad = 0.4352\,\pi R^3 \tau_{\max}$

$T_S \quad = \quad \tau_{\max} I_p / R \quad = \quad \tau_{\max} (0.5\,\pi R^4) / R \quad = \quad 0.5\,\pi R^3 \tau_{\max}$

$W_H \quad = \quad 0.64\,\pi R^2 L \gamma \qquad\qquad W_S \quad = \pi R^2 L \gamma$

thus $\quad S_H \quad = \quad T_H / W_H \quad = \quad 0.68\,\tau_{\max} R / \gamma L$

$S_S \quad = \quad T_S / W_S \quad = \quad 0.5\,\tau_{\max} R / \gamma L$

S_H is 36% greater than S_S

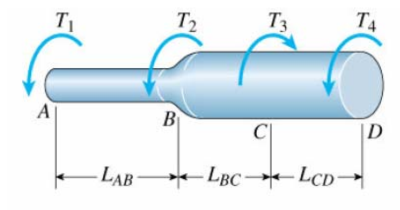
3.4 Nonuniform Torsion

(1) constant torque through each segment

$T_{CD} \quad = \quad -T_1 \quad - \quad T_2 \quad + \quad T_3$

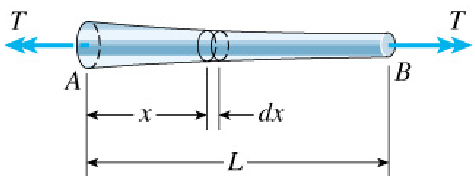
$T_{BC} \quad = \quad -T_1 \quad -T_2 \qquad T_{AB} \quad = \quad -T_1$

$$\phi \quad = \quad \sum_{i=1}^{n} \phi_i \quad = \quad \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_{pi}}$$



(a)

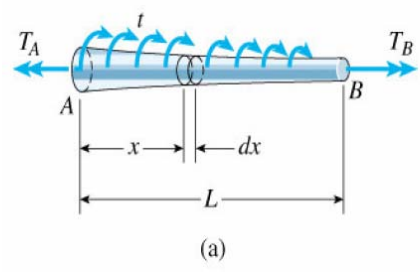
(2) constant torque with continuously varying cross section



$$d\phi = \frac{T\,dx}{G\,I_p(x)}$$

$$\phi = \int_0^L d\phi = \int_0^L \frac{T\,dx}{G\,I_p(x)}$$

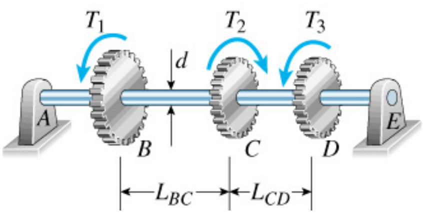
(3) continuously varying cross section and continuously varying torque



(a)

$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)\,dx}{G\,I_p(x)}$$

Example 3-4



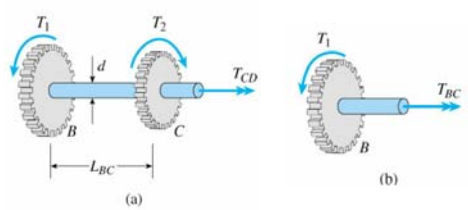
a solid steel shaft *ABCDE*, $d = 30$ mm

$T_1 = 275$ N-m $T_2 = 450$ N-m

$T_3 = 175$ N-m $G = 80$ GPa

$L_1 = 500$ mm $L_2 = 400$ mm

determine τ_{max} in each part and ϕ_{BD}



(a) (b)

$$T_{CD} = T_2 - T_1 = 175 \text{ N-m}$$

$$T_{BC} = -T_1 = -275 \text{ N-m}$$

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi\,d^3} = \frac{16 \times 275 \times 10^3}{\pi\,30^3} = 51.9 \text{ MPa}$$

$$\tau_{CD} = \frac{16\,T_{CD}}{\pi\,d^3} = \frac{16 \times 175 \times 10^3}{\pi\,30^3} = 33 \text{ MPa}$$

$$\phi_{BD} = \phi_{BC} + \phi_{CD}$$

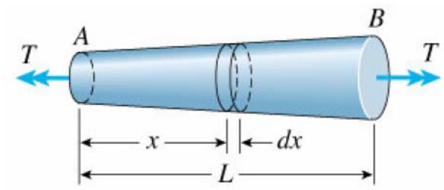
$$I_p = \frac{\pi\,d^4}{32} = \frac{\pi\,30^4}{32} = 79{,}520 \text{ mm}^2$$

$$\phi_{BC} = \frac{T_{BC} L_1}{G I_p} = \frac{-275 \times 10^3 \times 500}{80 \times 10^3 \times 79{,}520} = -0.0216 \text{ rad}$$

$$\phi_{CD} = \frac{T_{CD} L_2}{G I_p} = \frac{175 \times 10^3 \times 400}{80 \times 10^3 \times 79{,}520} = 0.011 \text{ rad}$$

$$\phi_{BD} = \phi_{BC} + \phi_{CD} = -0.0216 + 0.011 = -0.0106 \text{ rad} = -0.61^\circ$$

Example 3-5



a tapered bar AB of solid circular cross section is twisted by torque T

$d = d_A$ at A, $d = d_B$ at B, $d_B \geqq d_A$

determine τ_{max} and ϕ of the bar

(a) T = constant over the length,

thus τ_{max} occurs at d_{min} [end A]

$$\tau_{max} = \frac{16\,T}{\pi\, d_A^3}$$

(b) angle of twist

$$d(x) = d_A + \frac{d_B - d_A}{L} x$$

$$I_p(x) = \frac{\pi\, d^4}{32} = \frac{\pi}{32}\left(d_A + \frac{d_B - d_A}{L} x\right)^4$$

then

$$\phi = \int_0^L \frac{T\,dx}{G\, I_p(x)} = \frac{32\,T}{\pi\, G} \int_0^L \frac{dx}{\left(d_A + \frac{d_B - d_A}{L} x\right)^4}$$

to evaluate the integral, we note that it is of the form

$$\int \frac{dx}{(a + bx)^4} = - \frac{1}{3\, b\, (a + bx)^3}$$

if we choose $a = d_A$ and $b = (d_B - d_A) / L$, then the integral of ϕ can be obtained

$$\phi = \frac{32\, T\, L}{3\pi G (d_B - d_A)} \left(\frac{1}{d_A^3} - \frac{1}{d_B^3} \right)$$

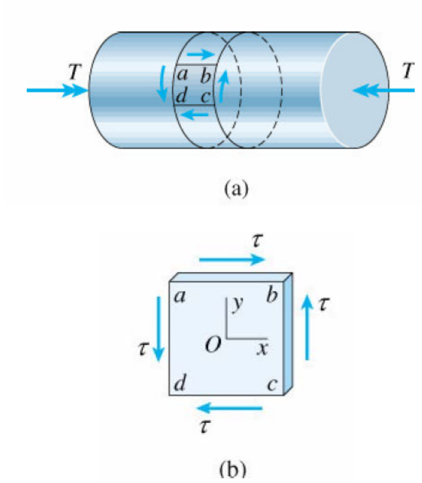
a convenient form can be written

$$\phi = \frac{T\, L}{G\, I_{pA}} \left(\frac{\beta^2 + \beta + 1}{3\, \beta^3} \right)$$

where $\beta = d_B / d_A$ $\quad I_{pA} = \pi\, d_A^4 / 32$

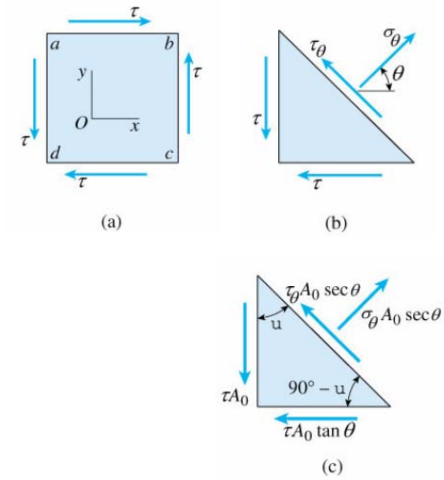
in the special case of a prismatic bar, $\beta = 1$, then $\phi = T\, L / G\, I_p$

3.5 Stresses and Strains in Pure Shear



for a circular bar subjected to torsion, shear stresses act over the cross sections and on longitudinal planes

an stress element *abcd* is cut between two cross sections and between two longitudinal planes, this element is in a state of pure shear



we now cut from the plane stress element to a wedge-shaped element, denote A_0 the area of the vertical side face, then the area of the bottom face is $A_0 \tan\theta$, and the area of the inclined face is A_0 sec θ

summing forces in the direction of σ_θ

$$\sigma_\theta A_0 \sec\theta \quad = \quad \tau A_0 \sin\theta \quad + \quad \tau A_0 \tan\theta \cos\theta$$

or $$\sigma_\theta \quad = \quad 2\,\tau \sin\theta \cos\theta \quad = \quad \tau \sin 2\theta$$

summing forces in the direction of τ_θ

$$\tau_\theta A_0 \sec\theta \quad = \quad \tau A_0 \cos\theta \quad - \quad \tau A_0 \tan\theta \sin\theta$$

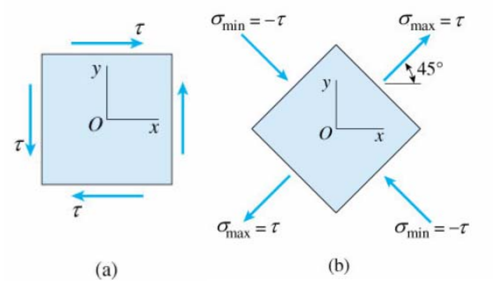
or $$\tau_\theta \quad = \quad \tau\,(\cos^2\theta \quad - \quad \sin^2\theta) \quad = \quad \tau \cos 2\theta$$

σ_θ and τ_θ vary with θ is plotted in figure

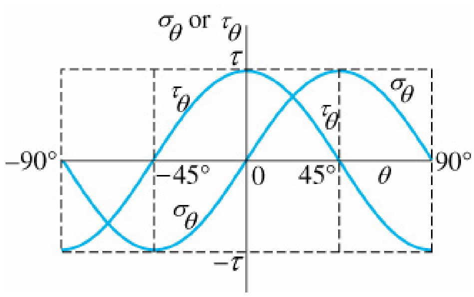
$(\tau_\theta)_{max} = \tau$ at $\theta = 0^o$

$(\tau_\theta)_{min} = -\,\tau$ at $\theta = \pm\,90^o$

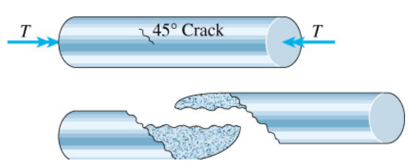
$(\sigma_\theta)_{max} = \pm\,\tau$ at $\theta = \pm\,45^o$



the state of pure shear stress is equivalent to equal tensile and compressive stresses on an element rotation through an angle of 45^o



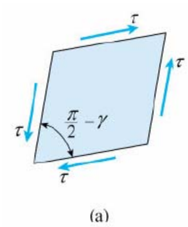
if a twisted bar is made of material that is weaker in tension than in shear, failure will occur in tension along a helix inclined at 45^o, such as chalk



Strains in pure shear

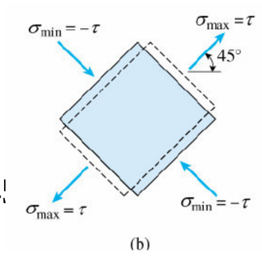
if the material is linearly elastic

$$\gamma \quad = \quad \tau \,/\, G$$



where G is the shear modulus of elasticity

consider the strains that occur in an element oriented at $\theta = 45^o$, $\sigma_{max} = \tau$ applied at 45^o and $\sigma_{min} = - \tau$ applied at $\theta = - 4$



then at $\theta = 45^o$

$$\varepsilon_{max} = \frac{\sigma_{max}}{E} - \frac{\nu\,\sigma_{min}}{E} = \frac{\tau}{E} + \frac{\nu\,\tau}{E} = \frac{\tau}{E}(1 + \nu)$$

at $\theta = - 45^o$ $\quad \varepsilon = - \varepsilon_{max} = - \tau\,(1 + \nu)\,/\,E$

it will be shown in next section the following relationship

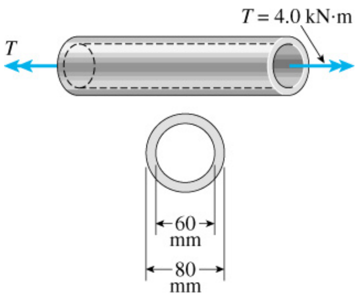
$$\varepsilon_{max} = \frac{\gamma}{2}$$

Example 3-6

a circular tube with d_o = 80 mm, d_i = 60 mm

T = 4 kN-m $\quad G$ = 27 GPa

determine (a) maximum tensile, compressive and shear stresses (b) maximum strains



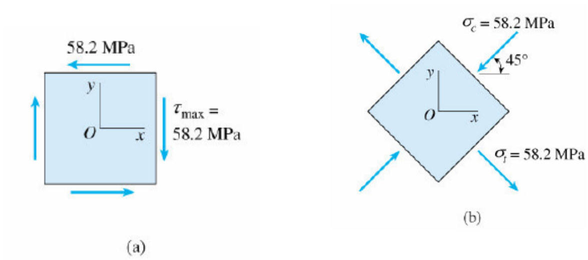
(a) the maximum shear stress is

$$\tau_{max} = \frac{T\,r}{I_p} = \frac{4000 \text{ x } 0.04}{\frac{\pi}{32}\,[(0.08)^4 - (0.06)^4]} = 58.2 \text{ MPa}$$

the maximum tensile and compressive stresses are

σ_t = 58.2 MPa $\quad$ at $\theta = - 45^o$

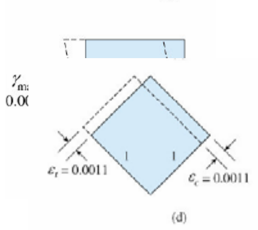
σ_c = - 58.2 MPa $\quad$ at $\theta = 45^o$



(b) maximum strains

$\gamma_{max} = \tau_{max} / G = 58.2 / 27 \times 10^3 = 0.0022$

the maximum normal strains is

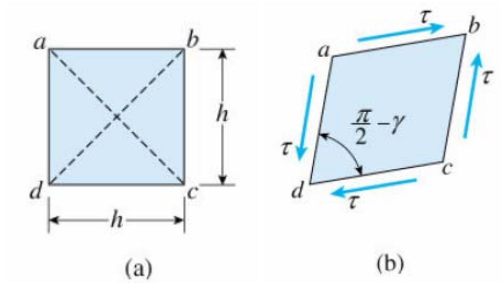


$\varepsilon_{max} = \gamma_{max} / 2 = 0.011$

i.e. $\varepsilon_t = 0.011 \qquad \varepsilon_c = -0.011$

3.6 Relationship Between Moduli of Elasticity *E*, *G* and *v*

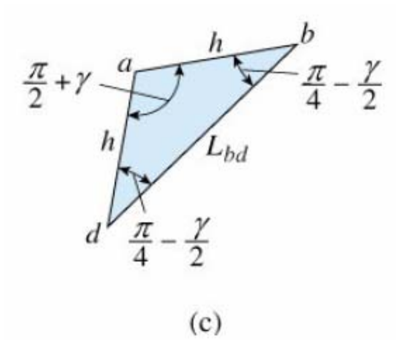
an important relationship between E, G and v can be obtained



consider the square stress element *abcd*, with the length of each side denoted as h, subjected to pure shear stress τ, then

$$\gamma = \tau / G$$

the length of diagonal bd is $\sqrt{2}\, h$, after deformation



$$L_{bd} = \sqrt{2}\, h\, (1 + \varepsilon_{max})$$

using the law of cosines for $\triangle\, abd$

$$L_{bd}^2 = h^2 + h^2 - 2\, h^2 \cos\left(\frac{\pi}{2} + \gamma\right) = 2\, h^2 \left[1 - \cos\left(\frac{\pi}{2} + \gamma\right)\right]$$

then $(1 + \varepsilon_{max})^2 = 1 - \cos\left(\frac{\pi}{2} + \gamma\right) = 1 + \sin\gamma$

thus $1 + 2\,\varepsilon_{max} + \varepsilon_{max}^2 = 1 + \sin\gamma$

$\because$ ε_{max} is very small, then $\varepsilon^2_{max} \rightarrow 0$, and $\sin\gamma \rightarrow \gamma$

the resulting expression can be obtained

$$\varepsilon_{max} = \gamma / 2$$

with $\varepsilon_{max} = \tau\,(1 + \nu) / E$ and $\gamma = \tau / G$

the following relationship can be written

$$G = \frac{E}{2\,(1 + \nu)}$$

thus E, G and ν are not independent properties of a linear elastic material

3.7 Transmission of Power by Circular Shafts

the most important use of circular shafts is to transmit mechanical power, such as drive shaft of an automobile, propeller shaft of a ship, axle of bicycle, torsional bar, etc.

a common design problem is the determination of the required size of a shaft so that it will transmit a specified amount of power at a specified speed of revolution without exceeding the allowable stress

consider a motor drive shaft, rotating at angular speed ω, it is transmitting a torque T, the work done is

$$W = T\,\phi \qquad \text{[T is constant for steady state]}$$

where ϕ is angular rotation in radians, ant the power is dW / dt

$$P = \frac{dW}{dt} = T\frac{d\phi}{dt} = T\omega \qquad \omega : \text{rad / s}$$

$$\because \quad \omega = 2\pi f \qquad f \text{ is frequency of revolution} \qquad f : \text{Hz} = \text{s}^{-1}$$

$$\therefore \quad P = 2\pi f T$$

denote n the number of revolution per minute (rpm), then $n = 60 f$

$$\text{thus} \quad P = \frac{2 n \pi T}{60} \qquad (n = \text{rpm},\ T = \text{N-m},\ P = \text{W})$$

in U.S. engineering practice, power is often expressed in horsepower (hp), 1 hp = 550 ft-lb / s, thus the horsepower H being transmitted by a rotating shaft is

$$H = \frac{2 n \pi T}{60 \times 550} = \frac{2 n \pi T}{33{,}000} \qquad (n = \text{rpm},\ T = \text{lb-ft},\ H = \text{hp})$$

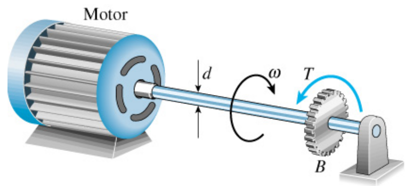
1 hp = 550 lb-ft/s = 550 x 4.448 N x 0.305 m/s = 746 N-m / s
= 746 W (W : watt)

Example 3-7

P = 30 kW, τ_{all} = 42 MPa

(a) n = 500 rpm, determine d

(b) n = 4000 rpm, determine d



(a) $$T = \frac{60 P}{2\pi n} = \frac{60 \times 30 \text{ kW}}{2\pi \times 500} = 573 \text{ N-m}$$

$$\tau_{max} = \frac{16 T}{\pi d^3} \qquad d^3 = \frac{16 T}{\pi \tau_{all}} = \frac{16 \times 573 \text{ N-m}}{\pi \times 42 \text{ MPa}} = 69.5 \times 10^{-6} \text{ m}^3$$

$$d = 41.1 \text{ mm}$$

(b)
$$T = \frac{60\,P}{2\,\pi\,n} = \frac{60 \text{ x } 30 \text{ kW}}{2\,\pi \text{ x } 4000} = 71.6 \text{ N-m}$$

$$d^3 = \frac{16\,T}{\pi\,\tau_{all}} = \frac{16 \text{ x } 71.6 \text{ N-m}}{\pi \text{ x } 42 \text{ MPa}} = 8.68 \text{ x } 10^{-6} \text{ m}^3$$

$$d = 20.55 \text{ mm}$$

the higher the speed of rotation, the smaller the required size of the shaft

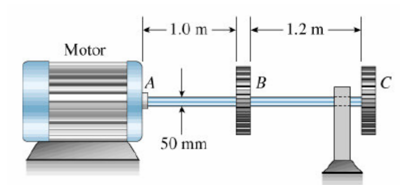
Example 3-8

a solid steel shaft ABC, $d = 50$ mm

motor A transmit 50 kW at 10 Hz

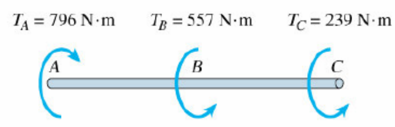
$P_B = 35$ kW, $P_C = 15$ kW

determine τ_{max} and ϕ_{AC}, $G = 80$ GPa



(a)

$$T_A = \frac{P_A}{2\,\pi\,f} = \frac{50 \text{ x } 10^3}{2\,\pi\,10} = 796 \text{ N-m}$$



(b)

similarly $P_B = 35$ kN $T_B = 557$ N-m

$$P_C = 15 \text{ kN} \qquad T_C = 239 \text{ N-m}$$

then $T_{AB} = 796$ N-m $T_{BC} = 239$ N-m

shear stress and angle of twist in segment AB

$$\tau_{AB} = \frac{16\,T_{AB}}{\pi\,d^3} = \frac{16 \text{ x } 796}{\pi\,50^3} = 32.4 \text{ MPa}$$

$$\phi_{AB} = \frac{T_{AB}\,L_{AB}}{G\,I_p} = \frac{796 \text{ x } 1.0}{80 \text{ x } 10^9 \frac{\pi}{32} 0.05^4} = 0.0162 \text{ rad}$$

shear stress and angle of twist in segment BC

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi\,d^3} = \frac{16 \text{ x } 239}{\pi\,50^3} = 9.7 \text{ MPa}$$

$$\phi_{AB} = \frac{T_{BC}\,L_{BC}}{G\,I_p} = \frac{239 \text{ x } 1.2}{80 \text{ x } 10^9\,\frac{\pi}{32}\,0.05^4} = 0.0058 \text{ rad}$$

$$\therefore \quad \tau_{max} = \tau_{AB} = 32.4 \text{ MPa}$$

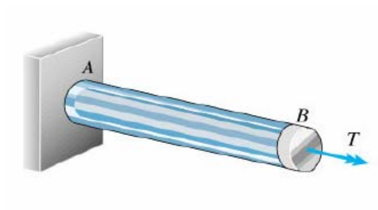
$$\phi_{AC} = \phi_{AB} + \phi_{BC} = 0.0162 + 0.0058 = 0.022 \text{ rad} = 1.26^{o}$$

3.8 Statically Indeterminate Torsional Members

torsional member may be statically indeterminate if they are constrained by more supports than are required to hold them in static equilibrium, or the torsional member is made by two or more kinds of materials

flexibility and stiffness methods may be used

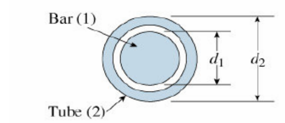
only flexibility method is used in the later discussion



(a)

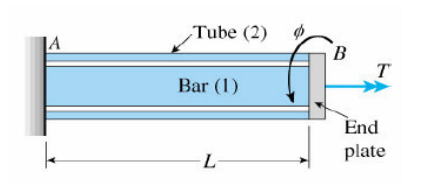
consider a composite bar AB fixed at A

the end plate rotates through an angle ϕ



(b)

T_1 and T_2 are developed in the solid bar and tube, respectively

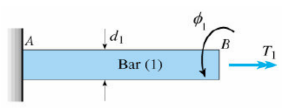


(c)

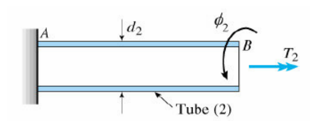
equation of equilibrium

$$T_1 + T_2 = T$$

equation of compatibility



(d)



(e)

$$\phi_1 = \phi_2$$

torque-displacement relations

$$\phi_1 = \frac{T_1 L}{G_1 I_{p1}} \qquad \phi_2 = \frac{T_2 L}{G_2 I_{p2}}$$

then the equation of compatibility becomes

$$\frac{T_1 L}{G_1 I_{p1}} = \frac{T_2 L}{G_2 I_{p2}}$$

now we can solve for T_1 and T_2

$$T_1 = T\left(\frac{G_1 I_{p1}}{G_1 I_{p1} + G_2 I_{p2}}\right) \quad T_2 = T\left(\frac{G_2 I_{p2}}{G_1 I_{p1} + G_2 I_{p2}}\right)$$

and

$$\phi = \frac{T L}{G_1 I_{p1} + G_2 I_{p2}}$$

Example 3-9

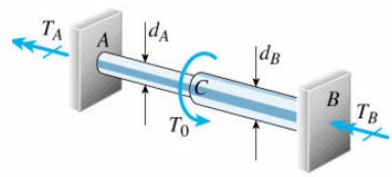
a bar ACB is fixed at both ends

T_0 is applied at point C

AC : d_A, L_A, I_{pA}

CB : d_B, L_B, I_{pB}

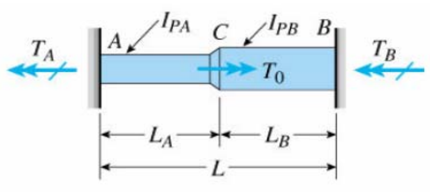
determine (a) T_A, T_B (b) τ_{AC}, τ_{CB} (c) ϕ_C



(a)

equation of equilibrium

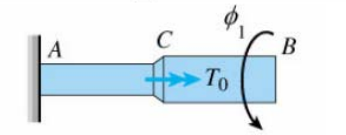
$$T_A + T_B = T_0$$



(b)

equation of compatibility

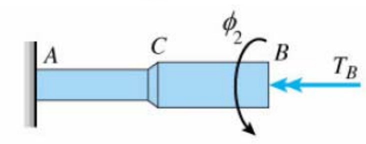
$$\phi_1 + \phi_2 = 0$$



(c)

torque-displacement equations

$$\phi_1 = T_0 L_A / G I_{pA}$$



(d)

$$\phi_2 = - \frac{T_B L_A}{G I_{pA}} - \frac{T_B L_B}{G I_{pB}}$$

then the equation of compatibility becomes

$$\frac{T_0 L_A}{G I_{pA}} - \frac{T_B L_A}{G I_{pA}} - \frac{T_B L_B}{G I_{pB}} = 0$$

T_A and T_B can be solved

$$T_A = T_0 \left(\frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} \right) \qquad T_B = T_0 \left(\frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} \right)$$

if the bar is prismatic, $I_{pA} = I_{pB} = I_p$

then $T_A = \dfrac{T_0 L_B}{L}$ $\qquad T_B = \dfrac{T_0 L_A}{L}$

maximum shear stress in AC and BC are

$$\tau_{AC} = \frac{T_A d_A}{2 I_{pA}} = \frac{T_0 L_B d_A}{2 (L_B I_{pA} + L_A I_{pB})}$$

$$\tau_{CB} = \frac{T_B d_B}{2 I_{pB}} = \frac{T_0 L_A d_B}{2 (L_B I_{pA} + L_A I_{pB})}$$

angle of rotation at section C is

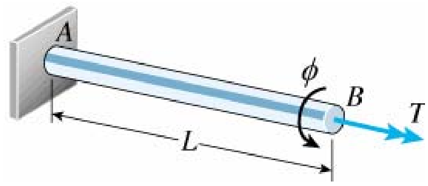
$$\phi_C = \frac{T_A L_A}{G I_{pA}} = \frac{T_B L_B}{G I_{pA}} = \frac{T_0 L_A L_B}{G (L_B I_{pA} + L_A I_{pB})}$$

if the bar is prismatic, $I_{pA} = I_{pB} = I_p$

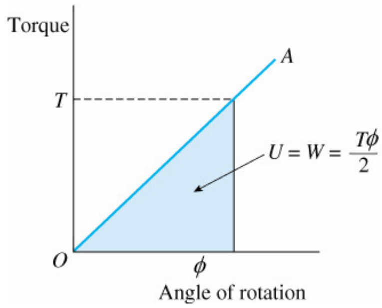
then $\phi_C = \dfrac{T_0 L_A L_B}{G L I_p}$

3.9 Strain Energy in Torsion and Pure Shear

consider a prismatic bar AB subjected to a torque T, the bar twists an angle ϕ



if the bar material is linear elastic, then the strain energy U of the bar is



$$U \;=\; W \;=\; T\,\phi\,/\,2$$

$$\because \qquad \phi \;=\; T\,L\,/\,G\,I_p$$

then
$$U \;=\; \frac{T^2\,L}{2\,G\,I_p} \;=\; \frac{G\,I_p\,\phi^2}{2\,L}$$

if the bar is subjected to nonuniform torsion, then

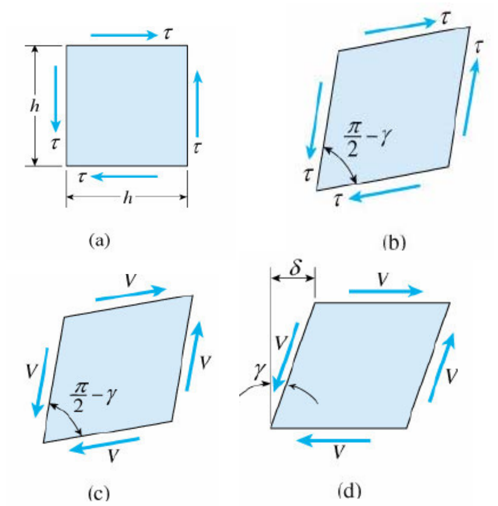
$$U \;=\; \sum_{i=1}^{n} U_i \;=\; \sum_{i=1}^{n} \frac{T_i^2\,L_i}{2\,G_i\,I_{pi}}$$

if either the cross section or the torque varies along the axis, then

$$dU \;=\; \frac{[T(x)]^2\,dx}{2\,G\,I_p(x)} \qquad U \;=\; \int dU \;=\; \int_0^L \frac{[T(x)]^2\,dx}{2\,G\,I_p(x)}$$

strain energy density in pure shear

consider a stressed element with each side having length h and thickness t, under shear stress τ with shear strain γ



the shear force V is

$$V \;=\; \tau\,h\,t$$

and the displacement δ is

$$\delta = h\,\gamma$$

for linear elastic material, strain energy stored in this element is

$$U = W = \frac{V\,\delta}{2} = \frac{\tau\,\gamma\,h^2\,t}{2}$$

and the strain energy density $u = U$ / per unit volume, then

$$u = \tau\,\gamma / 2 = \tau^2 / 2\,G = G\,\gamma^2 / 2$$

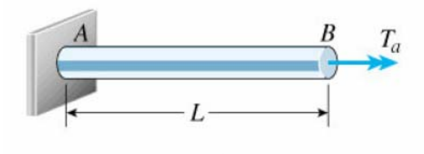
Example 3-10

a solid circular bar AB of length L

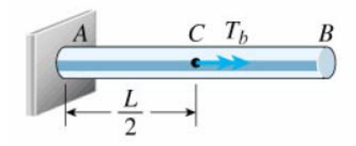
(a) torque T_a acting at the free end

(b) torque T_b acting at the midpoint

(c) both T_a and T_b acting simultaneously



(a)



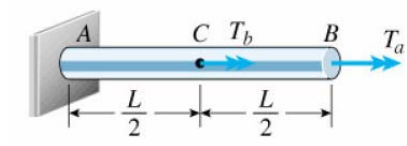
(b)

T_a = 100 N-m T_b = 150 N-m

L = 1.6 m G = 80 GPa

$I_p = 79.52 \times 10^3$ mm^4

determine the strain energy in each case



(c)

(a)

$$U_a = \frac{T_a^2\,L}{2\,G\,I_p} = \frac{100^2 \times 10^6 \times 1.6 \times 10^3}{2 \times 80 \times 10^3 \times 79.52 \times 10^3} = 1.26\ \text{J} \quad (\text{N-m})$$

(b)

$$U_b = \frac{T_b^2\,(L/2)}{2\,G\,I_p} = \frac{T_b^2\,L}{4\,G\,I_p} = 2.83\ \text{J}$$

(c)

$$U_c = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2 G_i I_{pi}} = \frac{T_a^2 (L/2)}{2 G I_p} + \frac{(T_a + T_b)^2 (L/2)}{2 G I_p}$$

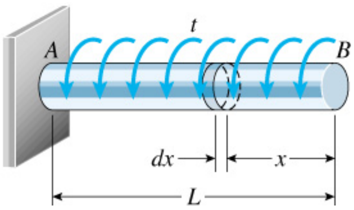
$$= \frac{T_a^2 L}{2 G I_p} + \frac{T_a T_b L}{2 G I_p} + \frac{T_b^2 L}{4 G I_p}$$

$$= 1.26 \text{ J} + 1.89 \text{ J} + 2.83 \text{ J} = 5.98 \text{ J}$$

Note that (c) is not equal to (a) + (b), because $U \sim T^2$

Example 3-11

a prismatic bar AB is loaded by a distributed torque of constant intensity t per unit distance



$t = 480$ lb-in/in $\quad L = 12$ ft

$G = 11.5 \times 10^6$ psi $\quad I_p = 18.17$ in^4

determine the strain energy

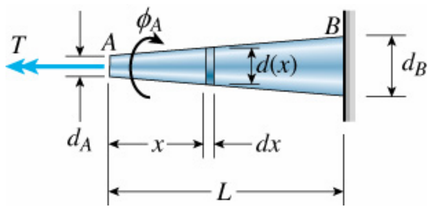
$$T(x) = t\,x$$

$$U = \int_0^L \frac{[(tx)]^2\,dx}{2 G I_p} = \frac{1}{2 G I_p} \int_0^L (tx)^2\,dx = \frac{t^2 L^3}{6 G I_p}$$

$$= \frac{480^2 \times (12 \times 12)^3}{6 \times 11.5 \times 10^6 \times 17.18} = 580 \text{ in-lb}$$

Example 3-12

a tapered bar AB of solid circular cross section is supported a torque T



$d = d_A \sim d_B$ from left to right

determine ϕ_A by energy method

$$W = \frac{T \phi_A}{2}$$

$$I_p(x) = \frac{\pi}{32} [d(x)]^4 = \frac{\pi}{32} (d_A + \frac{d_B - d_A}{L} x)^4$$

$$U = \int_0^L \frac{[T(x)]^2 dx}{2 G I_p(x)} = \frac{16 T^2}{\pi G} \int_0^L \frac{dx}{(d_A + \frac{d_B - d_A}{L} x)^4}$$

$$= \frac{16 T^2 L}{3 \pi G (d_B - d_A)} (\frac{1}{d_A^3} - \frac{1}{d_B^3})$$

with $U = W$, then ϕ_A can be obtained

$$\phi_A = \frac{32 T L}{3 \pi G (d_B - d_A)} (\frac{1}{d_A^3} - \frac{1}{d_B^3})$$

same result as in example 3-5

3-10 Thin-Walled Tubes

3-11 Stress Concentrations in Torsion

3-12 Nonlinear Torsion of Circular Bars