6S.1 Derivation of the Convection Transfer Equations

In Chapter 2 we considered a stationary substance in which heat is transferred by conduction and developed means for determining the temperature distribution within the substance. We did so by applying *conservation of energy* to a differential control volume (Figure 2.11) and deriving a differential equation that was termed the *heat equation*. For a prescribed geometry and boundary conditions, the equation may be solved to determine the corresponding temperature distribution.

If the substance is not stationary, conditions become more complex. For example, if conservation of energy is applied to a differential control volume in a moving fluid, the effects of fluid motion (*advection*) on energy transfer across the surfaces of the control volume must be considered, along with those of conduction. The resulting differential equation, which provides the basis for predicting the temperature distribution, now requires knowledge of the velocity field. This field must, in turn, be determined by solving additional differential equations derived by applying *conservation of mass* and *Newton's second law of motion* to a differential control volume.

In this supplemental material we consider conditions involving flow of a *viscous fluid* in which there is concurrent *heat* and *mass transfer*. Our objective is to develop differential equations that may be used to predict velocity, temperature, and species concentration fields within the fluid, and we do so by applying Newton's second law of motion and conservation of mass, energy, and species to a differential control volume. To simplify this development, we restrict our attention to *steady, two-dimensional flow* in the x and y directions of a Cartesian coordinate system. A unit depth may therefore be assigned to the z direction, thereby providing a differential control volume of extent $(dx \cdot dy \cdot 1)$.

6S.1.1 Conservation of Mass

One conservation law that is pertinent to the flow of a viscous fluid is that matter may neither be created nor destroyed. Stated in the context of the differential control volume of Figure 6S.1, this law requires that, for steady flow, *the net rate at which mass enters the control volume* (inflow − outflow) *must equal zero*. Mass enters and leaves the control volume exclusively through gross fluid motion. Transport due to such motion is often referred to as *advection*. If one corner of the control volume is located at (x, y), the rate at which mass enters the control volume through the surface perpendicular to x may be expressed as $(\rho u)\, dy$, where ρ is the total mass density ($\rho = \rho_A + \rho_B$) and u is the x component of the *mass average velocity*. The control volume is of unit depth in the z direction. Since ρ and u may vary with x, the

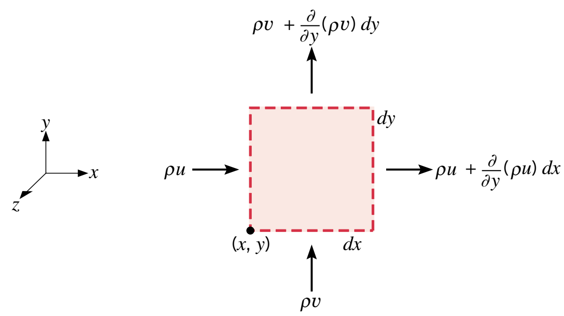


Figure 6S.1 Differential control volume $(dx \cdot dy \cdot 1)$ for mass conservation in two-dimensional flow of a viscous fluid.

rate at which mass leaves the surface at $x + dx$ may be expressed by a Taylor series expansion of the form

$$\left[(\rho u) + \frac{\partial(\rho u)}{\partial x}\, dx\right] dy$$

Using a similar result for the y direction, the conservation of mass requirement becomes

$$(\rho u)\, dy + (\rho v)\, dx - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\, dx\right] dy - \left[\rho v + \frac{\partial(\rho v)}{\partial y}\, dy\right] dx = 0$$

Canceling terms and dividing by $dx\, dy$, we obtain

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (6S.1)$$

Equation 6S.1, the *continuity equation*, is a general expression of the *overall mass* conservation requirement, and it must be satisfied at every point in the fluid. The equation applies for a single species fluid, as well as for mixtures in which species diffusion and chemical reactions may be occurring. If the fluid is *incompressible*, the density ρ is a constant, and the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (6S.2)$$

6S.1.2 Newton's Second Law of Motion

The second fundamental law that is pertinent to the flow of a viscous fluid is *Newton's second law of motion*. For a differential control volume in the fluid, this requirement states that the sum of all forces acting on the control volume must equal the net rate at which momentum leaves the control volume (outflow − inflow).

Two kinds of forces may act on the fluid: *body forces*, which are proportional to the volume, and *surface forces*, which are proportional to area. Gravitational, centrifugal, magnetic, and/or electric fields may contribute to the total body force, and we designate the x and y components of this force per unit volume of fluid as X and Y, respectively. The surface forces F_s are due to the fluid static pressure as well as to *viscous stresses*. At any point in the fluid, the viscous stress (a force per unit area)

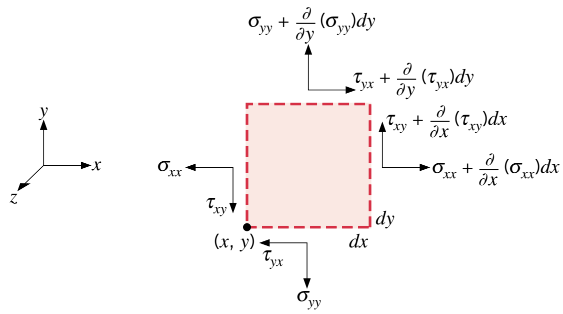


FIGURE 6S.2
Normal and shear viscous stresses for a differential control volume **$(dx \cdot dy \cdot 1)$** in two-dimensional flow of a viscous fluid.

may be resolved into two perpendicular components, which include a *normal stress* σ_{ii} and a *shear stress* τ_{ij} (Figure 6S.2).

A double subscript notation is used to specify the stress components. The first subscript indicates the surface orientation by providing the direction of its outward normal, and the second subscript indicates the direction of the force component. Accordingly, for the x surface of Figure 6S.2, the normal stress σ_{xx} corresponds to a force component normal to the surface, and the shear stress τ_{xy} corresponds to a force in the y direction along the surface. All the stress components shown are positive in the sense that *both* the surface normal and the force component are in the same direction. That is, they are both in either the positive coordinate direction or the negative coordinate direction. By this convention the normal viscous stresses are *tensile stresses*. In contrast the static pressure originates from an external force acting on the fluid in the control volume and is therefore a *compressive* stress.

Several features of the viscous stress should be noted. The associated force is between adjoining fluid elements and is a natural consequence of the fluid motion and viscosity. The surface forces of Figure 6S.2 are therefore presumed to act on the fluid within the control volume and are attributed to its interaction with the surrounding fluid. These stresses would vanish if the fluid velocity, or the velocity gradient, went to zero. In this respect the normal viscous stresses (σ_{xx} and σ_{yy}) must not be confused with the static pressure, which does not vanish for zero velocity.

Each of the stresses may change continuously in each of the coordinate directions. Using a Taylor series expansion for the stresses, the *net* surface force for each of the two directions may be expressed as

$$F_{s,x} = \left(\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) dx\, dy \qquad (6S.3)$$

$$F_{s,y} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial p}{\partial y}\right) dx\, dy \qquad (6S.4)$$

To use Newton's second law, the fluid momentum fluxes for the control volume must also be evaluated. If we focus on the x-direction, the relevant fluxes are as shown in Figure 6S.3. A contribution to the total x-momentum flux is made by the mass flow in each of the two directions. For example, the mass flux through the x surface (in the y-z plane) is (ρu), the corresponding x-momentum flux is $(\rho u)u$. Similarly, the x-momentum flux due to mass flow through the y surface (in the x-z

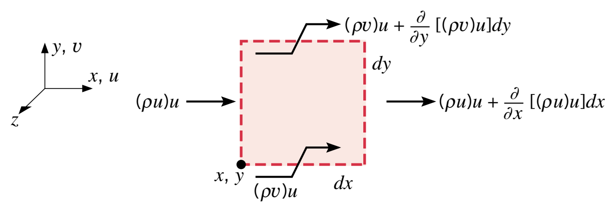


FIGURE 6S.3
Momentum fluxes for a differential control volume $(dx \cdot dy \cdot 1)$ in two-dimensional flow of a viscous fluid.

plane) is $(\rho v)u$. These fluxes may change in each of the coordinate directions, and the *net* rate at which x momentum leaves the control volume is

$$\frac{\partial[(\rho u)u]}{\partial x} dx\,(dy) + \frac{\partial[(\rho v)u]}{\partial y} dy\,(dx)$$

Equating the rate of change in the x momentum of the fluid to the sum of the forces in the x direction, we then obtain

$$\frac{\partial[(\rho u)u]}{\partial x} + \frac{\partial[(\rho v)u]}{\partial y} = \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + X \tag{6S.5}$$

This expression may be put in a more convenient form by expanding the derivatives on the left-hand side and substituting from the continuity equation, Equation 6S.1, giving

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\sigma_{xx} - p) + \frac{\partial \tau_{yx}}{\partial y} + X \tag{6S.6}$$

A similar expression may be obtained for the y direction and is of the form

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - p) + Y \tag{6S.7}$$

We should not lose sight of the physics represented by Equations 6S.6 and 6S.7. The two terms on the left-hand side of each equation represent the *net* rate of momentum flow from the control volume. The terms on the right-hand side account for the net viscous and pressure forces, as well as the body force. These equations must be satisfied at each point in the fluid, and with Equation 6S.1 they may be solved for the velocity field.

Before a solution to the foregoing equations can be obtained, it is necessary to relate the viscous stresses to other flow variables. These stresses are associated with the deformation of the fluid and are a function of the fluid viscosity and velocity gradients. From Figure 6S.4 it is evident that a *normal stress* must produce a *linear deformation* of the fluid, whereas a *shear stress* produces an *angular deformation*. Moreover, the magnitude of a stress is proportional to the *rate* at which the deformation occurs. The deformation rate is, in turn, related to the fluid viscosity and to the velocity gradients in the flow. For a *Newtonian fluid*[1] the stresses are proportional to

[1]A Newtonian fluid is one for which the shear stress is linearly proportional to the rate of angular deformation. All fluids of interest in the text are Newtonian.

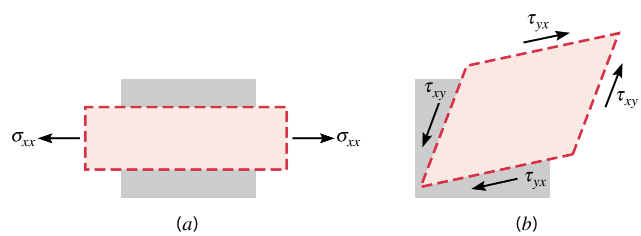


FIGURE 6S.4
Deformations of a fluid element due to viscous stresses. (*a*) Linear deformation due to a normal stress. (*b*) Angular deformation due to shear stresses.

the velocity gradients, where the proportionality constant is the fluid viscosity. Because of its complexity, however, development of the specific relations is left to the literature [1], and we limit ourselves to a presentation of the results. In particular, it has been shown that

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6S.8}$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6S.9}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6S.10}$$

Substituting Equations 6S.8 through 6S.10 into Equations 6S.6 and 6S.7, the x- and y-momentum equations become

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left\{\mu\left[2\frac{\partial u}{\partial x} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\right\} + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + X \tag{6S.11}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left\{\mu\left[2\frac{\partial v}{\partial y} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right]\right\} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + Y \tag{6S.12}$$

Equations 6S.1, 6S.11, and 6S.12 provide a complete representation of conditions in a two-dimensional viscous flow, and the corresponding velocity field may be determined by solving the equations. Once the velocity field is known, it is a simple matter to obtain the wall shear stress τ_s from Equation 6.2.

Equations 6S.11 and 6S.12 may be simplified for an *incompressible fluid* of *constant viscosity*. Rearranging the right-hand side of each expression and substituting from Equation 6S.2, the x- and y-momentum equations become

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X \tag{6S.13}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Y \tag{6S.14}$$

6S.1.3 Conservation of Energy

To apply the energy conservation requirement (Equation 1.11c) to a differential control volume in a viscous fluid with heat transfer (Figure 6S.5), it is necessary to first delineate the relevant physical processes. If potential energy effects are treated as work done by the body forces, the energy per unit mass of the fluid includes the thermal internal energy e and the kinetic energy $V^2/2$, where $V^2 \equiv u^2 + v^2$. Accordingly, thermal and kinetic energy are *advected* with the *bulk fluid* motion across the control surfaces, and for the x-direction, the *net* rate at which this energy *enters* the control volume is

$$\begin{aligned}\dot{E}_{\text{adv},x}-\dot{E}_{\text{adv},x+dx} &\equiv \rho u\left(e+\frac{V^2}{2}\right)dy-\left\{\rho u\left(e+\frac{V^2}{2}\right)\right.\\&\qquad\left.+\frac{\partial}{\partial x}\left[\rho u\left(e+\frac{V^2}{2}\right)\right]dx\right\}dy\\&=-\frac{\partial}{\partial x}\left[\rho u\left(e+\frac{V^2}{2}\right)\right]dx\,dy\end{aligned} \tag{6S.15}$$

Energy is also transferred across the control surface by *molecular processes*. There may be two contributions: that due to *conduction* and energy transfer due to the *diffusion of species* A and B. However, it is only in chemically reacting flows that species diffusion strongly influences thermal conditions. Hence the effect is neglected in this development. For the conduction process, the *net* transfer of energy into the control volume is

$$\begin{aligned}\dot{E}_{\text{cond},x}-\dot{E}_{\text{cond},x+dx} &= -\left(k\frac{\partial T}{\partial x}\right)dy-\left[-k\frac{\partial T}{\partial x}-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]dy\\&=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\,dy\end{aligned} \tag{6S.16}$$

Energy may also be transferred to and from the fluid in the control volume by *work* interactions involving the *body* and *surface forces*. The *net* rate at which work is done on the fluid by forces in the x-direction may be expressed as

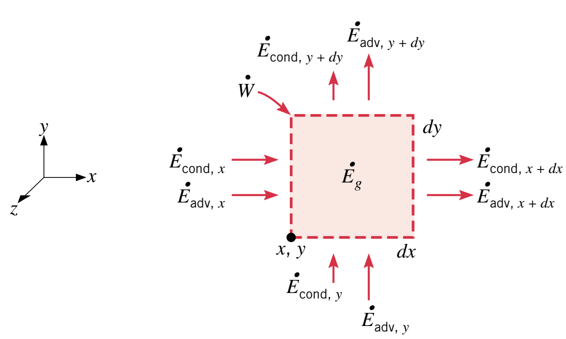


FIGURE 6S.5 Differential control volume ($dx \cdot dy \cdot 1$) for energy conservation in two-dimensional flow of a viscous fluid with heat transfer.

$$\dot{W}_{\text{net},x} = (Xu)\,dx\,dy + \frac{\partial}{\partial x}[(\sigma_{xx} - p)\,u]\,dx\,dy + \frac{\partial}{\partial y}(\tau_{yx}u)\,dx\,dy \tag{6S.17}$$

The first term on the right-hand side of Equation 6S.17 represents the work done by the body force, and the remaining terms account for the *net* work done by the pressure and viscous forces.

Using Equations 6S.15 through 6S.17, as well as analogous equations for the y-direction, the energy conservation requirement (Equation 1.11c) may be expressed as

$$\begin{aligned}
&-\frac{\partial}{\partial x}\left[\rho u\left(e + \frac{V^2}{2}\right)\right] - \frac{\partial}{\partial y}\left[\rho v\left(e + \frac{V^2}{2}\right)\right] \\
&+ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + (Xu + Yv) - \frac{\partial}{\partial x}(pu) - \frac{\partial}{\partial y}(pv) \\
&+ \frac{\partial}{\partial x}(\sigma_{xx}u + \tau_{xy}v) + \frac{\partial}{\partial y}(\tau_{yx}u + \sigma_{yy}v) + \dot{q} = 0
\end{aligned} \tag{6S.18}$$

where $\dot{q}$ is the rate at which thermal energy is generated per unit volume. This expression provides a general form of the energy conservation requirement for flow of a viscous fluid with heat transfer.

Because Equation 6S.18 represents conservation of *kinetic* and *thermal internal* energy, it is rarely used in solving heat transfer problems. Instead, a more convenient form, which is termed the *thermal energy equation*, is obtained by multiplying Equations 6S.6 and 6S.7 by u and v, respectively, and subtracting the results from Equation 6S.18. After considerable manipulation, it follows that [2]

$$\rho u\frac{\partial e}{\partial x} + \rho v\frac{\partial e}{\partial y} = \frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\Phi + \dot{q} \tag{6S.19}$$

where the term $p(\partial u/\partial x + \partial v/\partial y)$ represents a reversible conversion between mechanical work and thermal energy, and $\mu\Phi$, the *viscous dissipation*, is defined as

$$\mu\Phi \equiv \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2\right\} \tag{6S.20}$$

The first term on the right-hand side of Equation 6S.20 originates from the viscous shear stresses, and the remaining terms arise from the viscous normal stresses. Collectively, the terms account for the rate at which *mechanical work is irreversibly converted to thermal energy due to viscous effects in the fluid*.

If the fluid is incompressible, Equations 6S.19 and 6S.20 may be simplified by substituting Equation 6S.2. Moreover, with $de = c_v dT$ and $c_v = c_p$ for an incompressible fluid, the thermal energy equation may then be expressed as

$$\rho c_p\left(u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) + \mu\Phi + \dot{q} \tag{6S.21}$$

where

$$\mu\Phi = \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} \tag{6S.22}$$

The thermal energy equation may also be cast in terms of the fluid enthalpy i, instead of its internal energy e. Introducing the definition of the enthalpy,

$$i = e + \frac{p}{\rho} \tag{6S.23}$$

and using Equation 6S.1 to replace the third term on the right-hand side of Equation 6S.19 by spatial derivatives of p and (p/ρ), the energy equation may be expressed as [2]

$$\rho u \frac{\partial i}{\partial x} + \rho v \frac{\partial i}{\partial y} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + \mu\Phi + \dot{q} \tag{6S.24}$$

If the fluid may be approximated as a *perfect gas*, $di = c_p dT$, Equation 6S.24 becomes

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) + \mu\Phi + \dot{q} \tag{6S.25}$$

6S.1.4 *Conservation of Species*

If the viscous fluid consists of a binary mixture in which there are species concentration gradients (Figure 6.9), there will be *relative* transport of the species, and *species conservation* must be satisfied at each point in the fluid. The pertinent form of the conservation equation may be obtained by identifying the processes that affect the *transport* and *generation* of species A for a differential control volume in the fluid.

Consider the control volume of Figure 6S.6. Species A may be transported by *advection* (with the mean velocity of the mixture) and by *diffusion* (relative to the mean motion) in each of the coordinate directions. The concentration may also be affected by chemical reactions, and we designate the rate at which the mass of species A is generated per unit volume due to such reactions as $\dot{n}_A$.

The *net* rate at which species A *enters* the control volume due to *advection* in the x-direction is

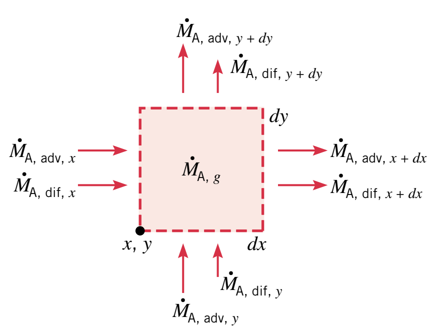


FIGURE 6S.6 Differential control volume $(dx \cdot dy \cdot 1)$ for species conservation in two-dimensional flow of a viscous fluid with mass transfer.

$$\dot{M}_{\text{A,adv},x} - \dot{M}_{\text{A,adv},x+dx} = (\rho_\text{A} u)\, dy - \left[(\rho_\text{A} u) + \frac{\partial(\rho_\text{A} u)}{\partial x} dx \right] dy$$
$$= -\frac{\partial(\rho_\text{A} u)}{\partial x} dx\, dy \tag{6S.26}$$

Similarly, multiplying both sides of Fick's law (Equation 6.6) by the molecular weight $\mathcal{M}_\text{A}$ (kg/kmol) of species A to evaluate the diffusion flux, the *net* rate at which species A *enters* the control volume due to *diffusion* in the x-direction is

$$\dot{M}_{\text{A,dif},x} - \dot{M}_{\text{A,dif},x+dx} = \left(-D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) dy - \left[\left(-D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) \right.$$
$$\left. + \frac{\partial}{\partial x} \left(-D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) dx \right] dy = \frac{\partial}{\partial x} \left(D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) dx\, dy \tag{6S.27}$$

Expressions similar to Equations 6S.26 and 6S.27 may be formulated for the y-direction.

Referring to Figure 6S.6, the species conservation requirement is

$$\dot{M}_{\text{A,adv},x} - \dot{M}_{\text{A,adv},x+dx} + \dot{M}_{\text{A,adv},y} - \dot{M}_{\text{A,adv},y+dy}$$
$$+ \dot{M}_{\text{A,dif},x} - \dot{M}_{\text{A,dif},x+dx} + \dot{M}_{\text{A,dif},y} - \dot{M}_{\text{A,dif},y+dy} + \dot{M}_{\text{A},g} = 0 \tag{6S.28}$$

Substituting from Equations 6S.26 and 6S.27, as well as from similar forms for the y-direction, it follows that

$$\frac{\partial(\rho_\text{A} u)}{\partial x} + \frac{\partial(\rho_\text{A} v)}{\partial y} = \frac{\partial}{\partial x} \left(D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_\text{AB} \frac{\partial \rho_\text{A}}{\partial y} \right) + \dot{n}_\text{A} \tag{6S.29}$$

A more useful form of this equation may be obtained by expanding the terms on the left-hand side and substituting from the overall continuity equation for an incompressible fluid. Equation 6S.29 then reduces to

$$u \frac{\partial \rho_\text{A}}{\partial x} + v \frac{\partial \rho_\text{A}}{\partial y} = \frac{\partial}{\partial x} \left(D_\text{AB} \frac{\partial \rho_\text{A}}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_\text{AB} \frac{\partial \rho_\text{A}}{\partial y} \right) + \dot{n}_\text{A} \tag{6S.30}$$

or in molar form

$$u \frac{\partial C_\text{A}}{\partial x} + v \frac{\partial C_\text{A}}{\partial y} = \frac{\partial}{\partial x} \left(D_\text{AB} \frac{\partial C_\text{A}}{\partial x} \right) + \frac{\partial}{\partial y} \left(D_\text{AB} \frac{\partial C_\text{A}}{\partial y} \right) + \dot{N}_\text{A} \tag{6S.31}$$

Example 6S.1

One of the few situations for which *exact* solutions to the convection transfer equations may be obtained involves what is termed *parallel flow*. In this case fluid motion is only in one direction. Consider a special case of parallel flow involving stationary and moving plates of infinite extent separated by a distance L, with the intervening space filled by an incompressible fluid. This situation is referred to as Couette flow and occurs, for example, in a journal bearing.

1. What is the appropriate form of the continuity equation (Equation D.1)?
2. Beginning with the momentum equation (Equation D.2), determine the velocity distribution between the plates.

3. Beginning with the energy equation (Equation D.4), determine the temperature distribution between the plates.
4. Consider conditions for which the fluid is engine oil with $L = 3$ mm. The speed of the moving plate is $U = 10$ m/s, and the temperatures of the stationary and moving plates are $T_0 = 10°C$ and $T_L = 30°C$, respectively. Calculate the heat flux to each of the plates and determine the maximum temperature in the oil.

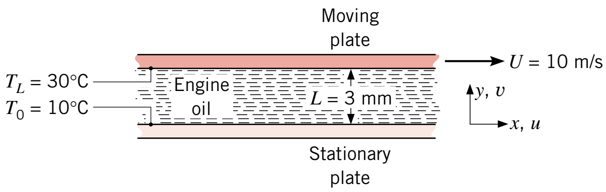
SOLUTION

Known: Couette flow with heat transfer.

Find:

1. Form of the continuity equation.
2. Velocity distribution.
3. Temperature distribution.
4. Surface heat fluxes and maximum temperature for prescribed conditions.

Schematic:



Assumptions:

1. Steady-state conditions.
2. Two-dimensional flow (no variations in z).
3. Incompressible fluid with constant properties.
4. No body forces.
5. No internal energy generation.

Properties: Table A.8, engine oil (20°C): $\rho = 888.2$ kg/m^3, $k = 0.145$ W/m · K, $\nu = 900 \times 10^{-6}$ m^2/s, $\mu = \nu\rho = 0.799$ N · s/m^2.

Analysis:

1. For an incompressible fluid (constant ρ) and parallel flow ($v = 0$), Equation D.1 reduces to

$$\frac{\partial u}{\partial x} = 0 \qquad \triangleleft$$

The important implication of this result is that, although depending on y, the x velocity component u is independent of x. It may then be said that the velocity field is *fully developed.*

2. For two-dimensional, steady-state conditions with $v = 0$, $(\partial u/\partial x) = 0$, and $X = 0$, Equation D.2 reduces to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

However, in Couette flow, motion of the fluid is not sustained by the pressure gradient, $\partial p/\partial x$, but by an external force that provides for motion of the top plate relative to the bottom plate. Hence $(\partial p/\partial x) = 0$. Accordingly, the x-momentum equation reduces to

$$\frac{\partial^2 u}{\partial y^2} = 0$$

The desired velocity distribution may be obtained by solving this equation. Integrating twice, we obtain

$$u(y) = C_1 y + C_2$$

where C_1 and C_2 are the constants of integration. Applying the boundary conditions

$$u(0) = 0 \qquad u(L) = U$$

it follows that $C_2 = 0$ and $C_1 = U/L$. The velocity distribution is then

$$u(y) = \frac{y}{L} U \qquad \triangleleft$$

3. The energy equation (D.4) may be simplified for the prescribed conditions. In particular, with $v = 0$, $(\partial u/\partial x) = 0$, and $\dot{q} = 0$, it follows that

$$\rho c_p u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

However, because the top and bottom plates are at uniform temperatures, the temperature field must also be fully developed, in which case $(\partial T/\partial x) = 0$. The appropriate form of the energy equation is then

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

The desired temperature distribution may be obtained by solving this equation. Rearranging and substituting for the velocity distribution,

$$k \frac{d^2 T}{dy^2} = -\mu \left(\frac{du}{dy}\right)^2 = -\mu \left(\frac{U}{L}\right)^2$$

Integrating twice, we obtain

$$T(y) = -\frac{\mu}{2k} \left(\frac{U}{L}\right)^2 y^2 + C_3 y + C_4$$

The constants of integration may be obtained from the boundary conditions

$$T(0) = T_0 \qquad T(L) = T_L$$

in which case

$$C_4 = T_0 \qquad \text{and} \qquad C_3 = \frac{T_L - T_0}{L} + \frac{\mu}{2k} \frac{U^2}{L}$$

and

$$T(y) = T_0 + \frac{\mu}{2k} U^2 \left[\frac{y}{L} - \left(\frac{y}{L}\right)^2\right] + (T_L - T_0) \frac{y}{L} \qquad \triangleleft$$

4. Knowing the temperature distribution, the surface heat fluxes may be obtained by applying Fourier's law. Hence

$$q''_y = -k\frac{dT}{dy} = -k\left[\frac{\mu}{2k}U^2\left(\frac{1}{L} - \frac{2y}{L^2}\right) + \frac{T_L - T_0}{L}\right]$$

At the bottom and top surfaces, respectively, it follows that

$$q''_0 = -\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_0) \qquad \text{and} \qquad q''_L = +\frac{\mu U^2}{2L} - \frac{k}{L}(T_L - T_0)$$

Hence, for the prescribed numerical values,

$$q''_0 = -\frac{0.799\ \text{N}\cdot\text{s/m}^2 \times 100\ \text{m}^2/\text{s}^2}{2\times 3\times 10^{-3}\text{m}} - \frac{0.145\ \text{W/m}\cdot\text{K}}{3\times 10^{-3}\text{m}}(30 - 10)°\text{C}$$

$$q''_0 = -13{,}315\ \text{W/m}^2 - 967\ \text{W/m}^2 = -14.3\ \text{kW/m}^2 \quad \triangleleft$$

$$q''_L = +13{,}315\ \text{W/m}^2 - 967\ \text{W/m}^2 = 12.3\ \text{kW/m}^2 \quad \triangleleft$$

The location of the maximum temperature in the oil may be found from the requirement that

$$\frac{dT}{dy} = \frac{\mu}{2k}U^2\left(\frac{1}{L} - \frac{2y}{L^2}\right) + \frac{T_L - T_0}{L} = 0$$

Solving for y, it follows that

$$y_{\max} = \left[\frac{k}{\mu U^2}(T_L - T_0) + \frac{1}{2}\right]L$$

or for the prescribed conditions

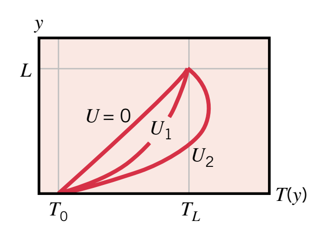
$$y_{\max} = \left[\frac{0.145\ \text{W/m}\cdot\text{K}}{0.799\ \text{N}\cdot\text{s/m}^2\times 100\ \text{m}^2/\text{s}^2}(30 - 10)°\text{C} + \frac{1}{2}\right]L = 0.536L$$

Substituting the value of $y_{\max}$ into the expression for $T(y)$,

$$T_{\max} = 89.2°\text{C} \quad \triangleleft$$

Comments:

1. Given the strong effect of viscous dissipation for the prescribed conditions, the maximum temperature occurs in the oil and there is heat transfer to the hot, as well as to the cold, plate. The temperature distribution is a function of the velocity of the moving plate, and the effect is shown schematically below.



For velocities less than U_1 the maximum temperature corresponds to that of the hot plate. For $U = 0$ there is no viscous dissipation, and the temperature distribution is linear.

2. Recognize that the properties were evaluated at $\overline{T} = (T_L + T_0)/2 = 20°C$, which is *not* a good measure of the average oil temperature. For more precise calculations, the properties should be evaluated at a more appropriate value of the average temperature (e.g., $\overline{T} \approx 55°C$), and the calculations should be repeated.

References

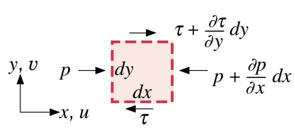
1. Schlichting, H., *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York, 1979.

2. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1966.

Problems

Conservation Equations and Solutions

6S.1 Consider the control volume shown for the special case of steady-state conditions with $v = 0$, $T = T(y)$, and ρ = const.



(a) Prove that $u = u(y)$ if $v = 0$ everywhere.

(b) Derive the x-momentum equation and simplify it as much as possible.

(c) Derive the energy equation and simplify it as much as possible.

6S.2 Consider a lightly loaded journal bearing using oil having the constant properties $\mu = 10^{-2}$ kg/s · m and $k = 0.15$ W/m · K. If the journal and the bearing are each maintained at a temperature of 40°C, what is the maximum temperature in the oil when the journal is rotating at 10 m/s?

6S.3 Consider a lightly loaded journal bearing using oil having the constant properties $\rho = 800$ kg/m^3, $\nu = 10^{-5}$ m^2/s, and $k = 0.13$ W/m · K. The journal diameter is 75 mm; the clearance is 0.25 mm; and the bearing operates at 3600 rpm.

(a) Determine the temperature distribution in the oil film assuming that there is no heat transfer into the journal and that the bearing surface is maintained at 75°C.

(b) What is the rate of heat transfer from the bearing, and how much power is needed to rotate the journal?

6S.4 Consider two large (infinite) parallel plates, 5 mm apart. One plate is stationary, while the other plate is moving at a speed of 200 m/s. Both plates are maintained at 27°C. Consider two cases, one for which the plates are separated by water and the other for which the plates are separated by air.

(a) For each of the two fluids, what is the force per unit surface area required to maintain the above condition? What is the corresponding power requirement?

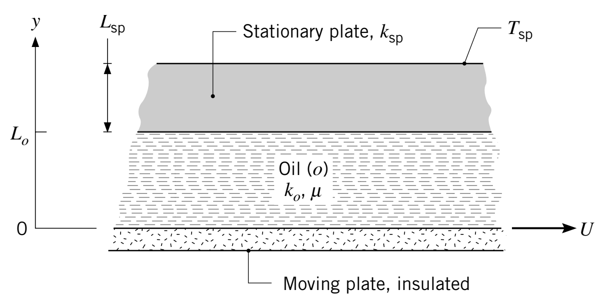
(b) What is the viscous dissipation associated with each of the two fluids?

(c) What is the maximum temperature in each of the two fluids?

6S.5 A judgment concerning the influence of viscous dissipation in forced convection heat transfer may be made by calculating the quantity $Pr \cdot Ec$, where the Prandtl number $Pr = c_p\mu/k$ and the Eckert number $Ec = U^2/c_p\,\Delta T$ are *dimensionless groups.* The characteristic velocity and temperature difference of the problem are designated as U and ΔT, respectively. If $Pr \cdot Ec \ll 1$, dissipation effects may be neglected. Consider Couette flow for which one plate moves at 10 m/s and a temperature difference of 25°C is maintained between the plates. Evaluating properties at 27°C, determine the value of $Pr \cdot Ec$ for air, water, and engine oil. What is the value of $Pr \cdot Ec$ for air if the plate is moving at the sonic velocity?

6S.6 Consider Couette flow for which the moving plate is maintained at a uniform temperature and the stationary plate is insulated. Determine the temperature of the insulated plate, expressing your result in terms of fluid properties and the temperature and speed of the moving plate. Obtain an expression for the heat flux at the moving plate.

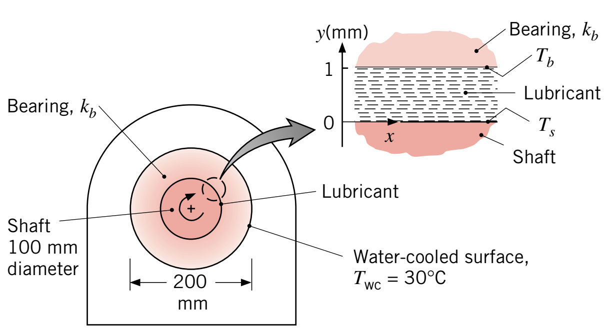
6S.7 Consider Couette flow with heat transfer for which the lower plate (mp) moves with a speed of $U = 5$ m/s and is perfectly insulated. The upper plate (sp) is stationary and is made of a material with thermal conductivity $k_{sp} = 1.5$ W/m · K and thickness $L_{sp} = 3$ mm. Its outer surface is maintained at $T_{sp} = 40°C$. The plates are separated by a distance $L_o = 5$ mm, which is filled with an engine oil of viscosity $\mu = 0.799$ N · s/m^2 and thermal conductivity $k_o = 0.145$ W/m · K.



(a) On $T(y)$–y coordinates, sketch the temperature distribution in the oil film and the moving plate.

(b) Obtain an expression for the temperature at the lower surface of the oil film, $T(0) = T_o$, in terms of the plate speed U, the stationary plate parameters (T_{sp}, k_{sp}, L_{sp}) and the oil parameters (μ, k_o, L_o). Calculate this temperature for the prescribed conditions.

6S.8 A shaft with a diameter of 100 mm rotates at 9000 rpm in a journal bearing that is 70 mm long. A uniform lubricant gap of 1 mm separates the shaft and the bearing. The lubricant properties are $\mu = 0.03$ N · s/m^2 and $k = 0.15$ W/m · K, while the bearing material has a thermal conductivity of $k_b = 45$ W/m · K.



(a) Determine the viscous dissipation, $\mu\Phi$ (W/m^3), in the lubricant.

(b) Determine the rate of heat transfer (W) from the lubricant, assuming that no heat is lost through the shaft.

(c) If the bearing housing is water-cooled, such that the outer surface of the bearing is maintained at 30°C, determine the temperatures of the bearing and shaft, T_b and T_s.

6S.9 Consider Couette flow with heat transfer as described in Example 6S.1.

(a) Rearrange the temperature distribution to obtain the dimensionless form

$$\theta(\eta) = \eta[1 + \tfrac{1}{2} Pr\, Ec(1 - \eta)]$$

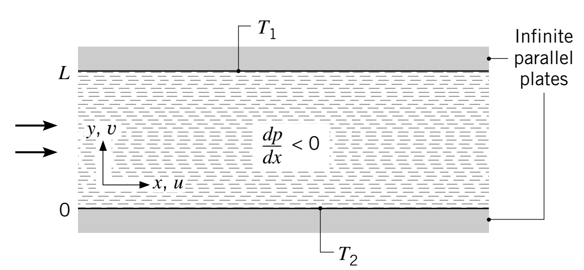
where $\theta \equiv [T(y) - T_0]/[T_L - T_0]$ and $\eta = y/L$. The *dimensionless groups* are the Prandtl number $Pr = \mu c_p/k$ and the Eckert number $Ec = U^2/c_p(T_L - T_0)$.

(b) Derive an expression that prescribes the conditions under which there will be no heat transfer to the upper plate.

(c) Derive an expression for the heat transfer rate to the lower plate for the conditions identified in part (b).

$\boxed{\text{(d)}}$ Generate a plot of θ versus η for $0 \leq \eta \leq 1$ and values of $Pr\, Ec = 0, 1, 2, 4, 10$. Explain key features of the temperature distributions.

6S.10 Consider the problem of steady, incompressible laminar flow between two stationary, infinite parallel plates maintained at different temperatures.



Referred to as *Poiseuille flow* with heat transfer, this special case of parallel flow is one for which the x velocity component is finite, but the y and z components (v and w) are zero.

(a) What is the form of the continuity equation for this case? In what way is the flow *fully developed*?

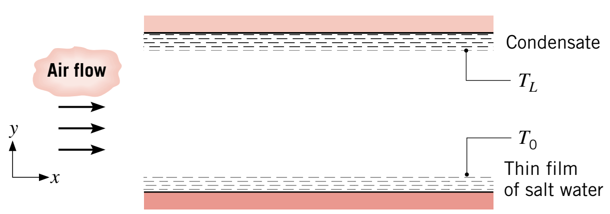
(b) What forms do the x- and y-momentum equations take? What is the form of the velocity profile? Note that, unlike Couette flow, fluid motion between the plates is now sustained by a finite pressure gradient. How is this pressure gradient related to the maximum fluid velocity?

(c) Assuming viscous dissipation to be significant and recognizing that conditions must be thermally fully developed, what is the appropriate form of the energy equation? Solve this equation for the temperature distribution. What is the heat flux at the upper ($y = L$) surface?

Species Conservation Equation and Solution

6S.11 Consider Problem 6S.10, when the fluid is a binary mixture with different molar concentrations $C_{A,1}$ and $C_{A,2}$ at the top and bottom surfaces, respectively. For the region between the plates, what is the appropriate form of the species A continuity equation? Obtain expressions for the species concentration distribution and the species flux at the upper surface.

6S.12 A simple scheme for desalination involves maintaining a thin film of saltwater on the lower surface of two large (infinite) parallel plates that are slightly inclined and separated by a distance L.



A slow, incompressible, laminar airflow exists between the plates, such that the x velocity component is finite while the y and z components are zero. Evaporation occurs from the liquid film on the lower surface, which is maintained at an elevated temperature T_0, while condensation occurs at the upper surface, which is maintained at a reduced temperature T_L. The corresponding molar concentrations of water vapor at the lower and upper surfaces are designated as $C_{A,0}$ and $C_{A,L}$, respectively. The species concentration and temperature may be assumed to be independent of x and z.

(a) Obtain an expression for the distribution of the water vapor molar concentration $C_A(y)$ in the air. What is the mass rate of pure water production per unit surface area? Express your results in terms of $C_{A,0}$, $C_{A,L}$, L, and the vapor–air diffusion coefficient D_{AB}.

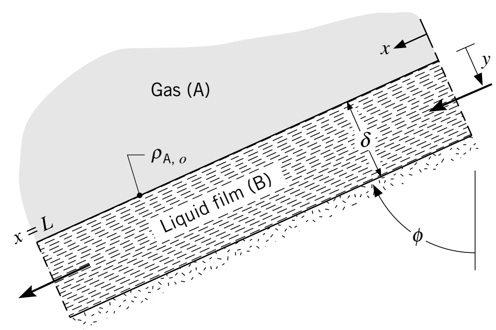
(b) Obtain an expression for the rate at which heat must be supplied per unit area to maintain the lower surface at T_0. Express your result in terms of $C_{A,0}$, $C_{A,L}$, T_0, T_L, L, D_{AB}, h_{fg} (the latent heat of vaporization of water), and the thermal conductivity k.

6S.13 Consider the conservation equations (6S.24) and (6S.31).

(a) Describe the physical significance of each term.

(b) Identify the approximations and special conditions needed to reduce these expressions to the boundary layer equations (6.29 and 6.30). Comparing these equations, identify the conditions under which they have the same form. Comment on the existence of a heat and mass transfer analogy.

6S.14 The *falling film* is widely used in chemical processing for the removal of gaseous species. It involves the flow of a liquid along a surface that may be inclined at some angle $\phi \geq 0$.



The flow is sustained by gravity, and the gas species A outside the film is absorbed at the liquid–gas interface. The film is in fully developed laminar flow over the entire plate, such that its velocity components in the y and z directions are zero. The mass density of A at $y = 0$ in the liquid is a constant $\rho_{A,o}$ independent of x.

(a) Write the appropriate form of the x-momentum equation for the film. Solve this equation for the distribution of the x velocity component, $u(y)$, in the film. Express your result in terms of δ, g, ϕ, and the liquid properties μ and ρ. Write an expression for the maximum velocity u_{max}.

(b) Obtain an appropriate form of the A species conservation equation for conditions within the film. If it is further assumed that the transport of species A across the gas–liquid interface does not penetrate very far into the film, the position $y = \delta$ may, for all practical purposes, be viewed as $y = \infty$. This condition implies that to a good approximation, $u = u_{max}$ in the region of penetration. Subject to these assumptions, determine an expression for $\rho_A(x, y)$ that applies in the film. *Hint:* This problem is analogous to conduction in a semi-infinite medium with a sudden change in surface temperature.

(c) If a local mass transfer convection coefficient is defined as

$$h_{m,x} \equiv \frac{n''_{A,x}}{\rho_{A,o}}$$

where $n''_{A,x}$ is the local mass flux at the gas–liquid interface, develop a suitable correlation for Sh_x as a function of Re_x and Sc.

(d) Develop an expression for the total gas absorption rate per unit width for a film of length L (kg/s · m).

(e) A water film that is 1 mm thick runs down the inside surface of a vertical tube that is 2 m long and has an inside diameter of 50 mm. An airstream containing ammonia (NH_3) moves through the tube, such that the mass density of NH_3 at the gas–liquid interface (but in the liquid) is 25 kg/m^3. A dilute solution of ammonia in water is formed, and the diffusion coefficient is 2×10^{-9} m^2/s. What is the mass rate of NH_3 removal by absorption?