Chapter 5

Time-Dependent Conduction

5.1 The Lumped Capacitance Method

This method assumes *spatially uniform* solid temperature at any instant during the transient process. It is valid if the temperature gradients within the solid are small.

Apply energy balance to the control volume of Fig. 5.1

$$-\dot{E}_{\text{out}} = \dot{E}_{\text{st}} \qquad (5.1)$$

If the energy exchange is through convection,

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt} \qquad (5.2)$$

Letting $\theta = T\text{-}T_\infty$, we obtain

$$\frac{\rho V c}{hA_s} \frac{d\theta}{dt} = -\theta$$

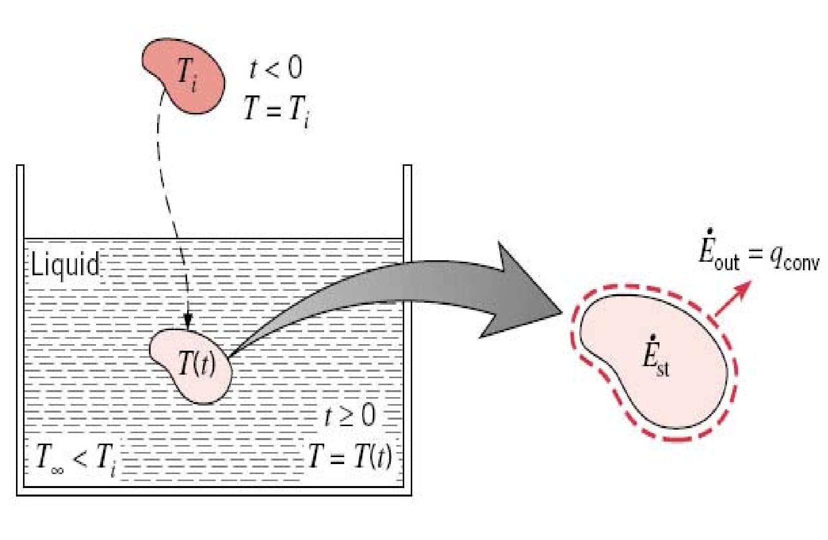


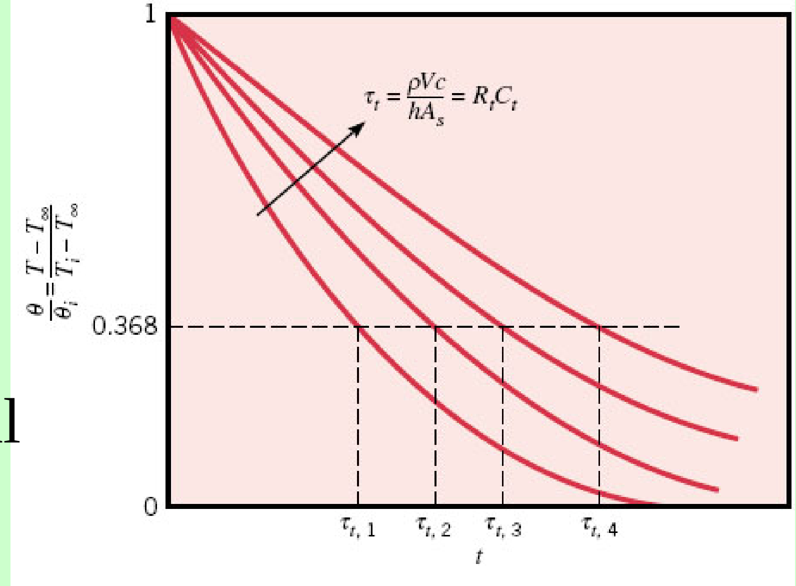
FIGURE 5.1 Cooling of a hot metal forging.

With the initial condition $\theta(0)=\theta_i$, it follows

$$\frac{\theta}{\theta_i}=\frac{T-T_\infty}{T_i-T_\infty}=\exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right] \qquad (5.6)$$

The *thermal time constant* is defined as

$$\tau_t=\frac{\rho Vc}{hA_s}=\left(\frac{1}{hA_s}\right)(\rho Vc)=R_tC_t$$



where R_t is the resistance to convection heat transfer, C_t is the lumped thermal capacitance of the solid.

The total energy transfer Q can be obtained by

$$Q=\int_0^t q\,dt=hA_s\int_0^t \theta\,dt=(\rho Vc)\theta_i\left[1-\exp\left(-\frac{t}{\tau_t}\right)\right] \qquad (5.8)$$

5.2 Validity of the Lumped Capacitance Method

To find the criterion for the validity of the lumped capacitance method, consider the steady-state energy balance (Fig. 5.3)

$$\frac{kA}{L}(T_{s,1}-T_{s,2}) = hA(T_{s,2}-T_{\infty})$$

k--conductivity of the solid

$$\rightarrow \frac{T_{s,1}-T_{s,2}}{T_{s,2}-T_{\infty}} = \frac{(L/kA)}{(1/hA)} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} \equiv Bi \qquad (5.9)$$

Criterion for the validity of the lumped capacitance method(Fig.5.4):

$$Bi = \frac{hL_c}{k} < 0.1 \qquad (5.10)$$

L_c= characteristic length

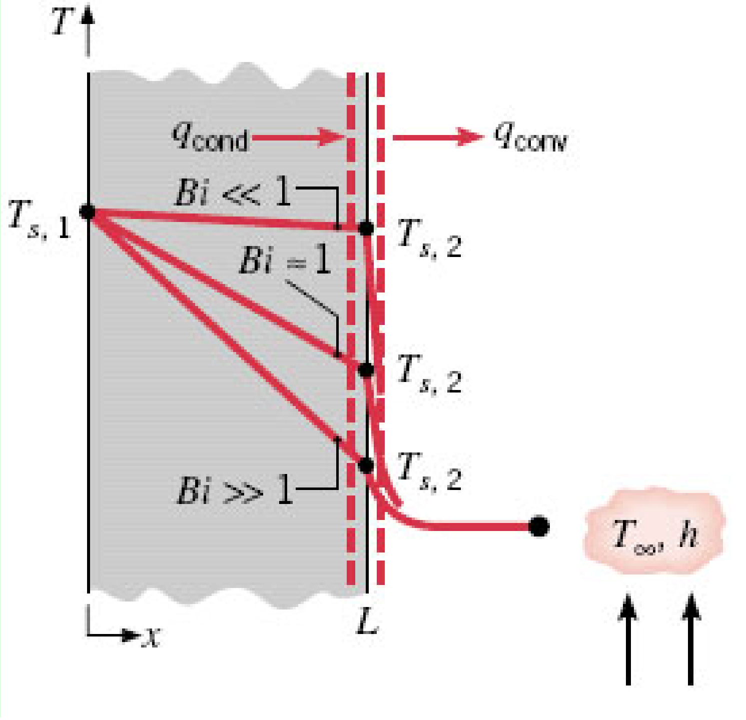


FIGURE 5.3
Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

Criterion for the validity of the lumped capacitance method (Fig.5.4): $Bi = \frac{hL_c}{k} < 0.1$ L_c= characteristic length

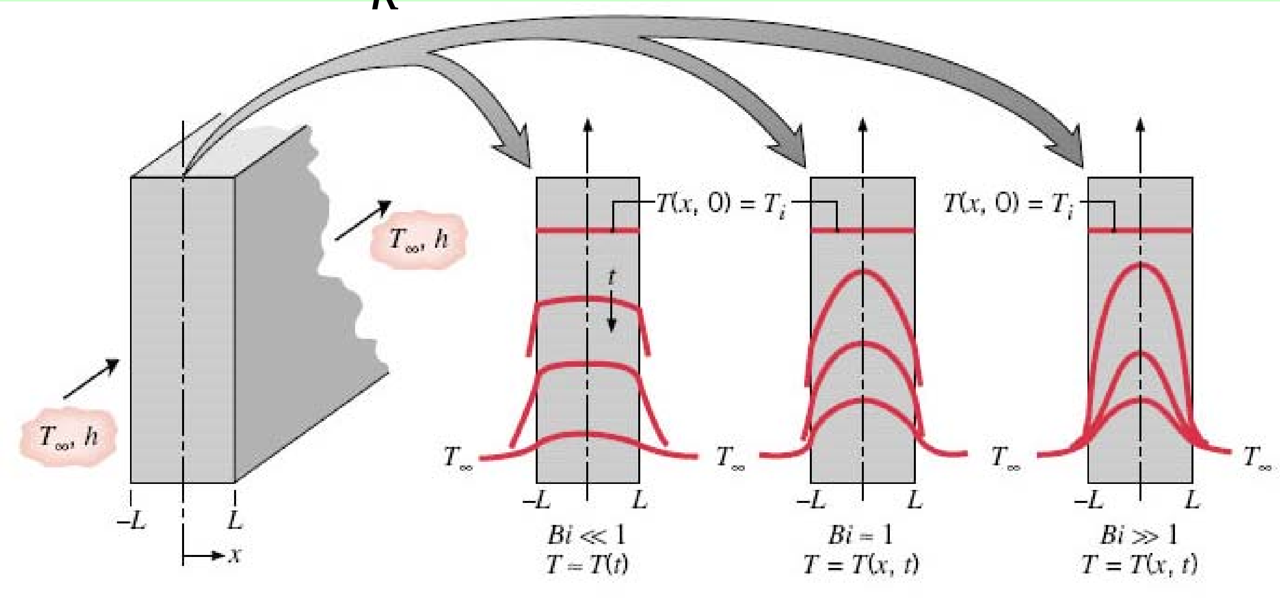


FIGURE 5.4 Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

The *dimensionless time*, Fourier number, $Fo = \frac{\alpha t}{L_c^2}$ (5.12)

Fo, with *Bi*, characterizes transient conduction problems.

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo) \quad (5.13)$$

EX 5.1

5.3 General Lumped Capacitance Analysis

A general situation including convection, radiation, an applied surface heat flux, and internal energy generation (Fig.5.5) can be described as

$$q_s'' A_{s,h} + \dot{E}_g - (q''_{\text{conv}} + q''_{\text{rad}}) A_{s(c,r)} = \rho V c \frac{dT}{dt} \qquad (5.14)$$

or $$q_s'' A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{\text{sur}}^4)] A_{s(c,r)} = \rho V c \frac{dT}{dt} \qquad (5.15)$$

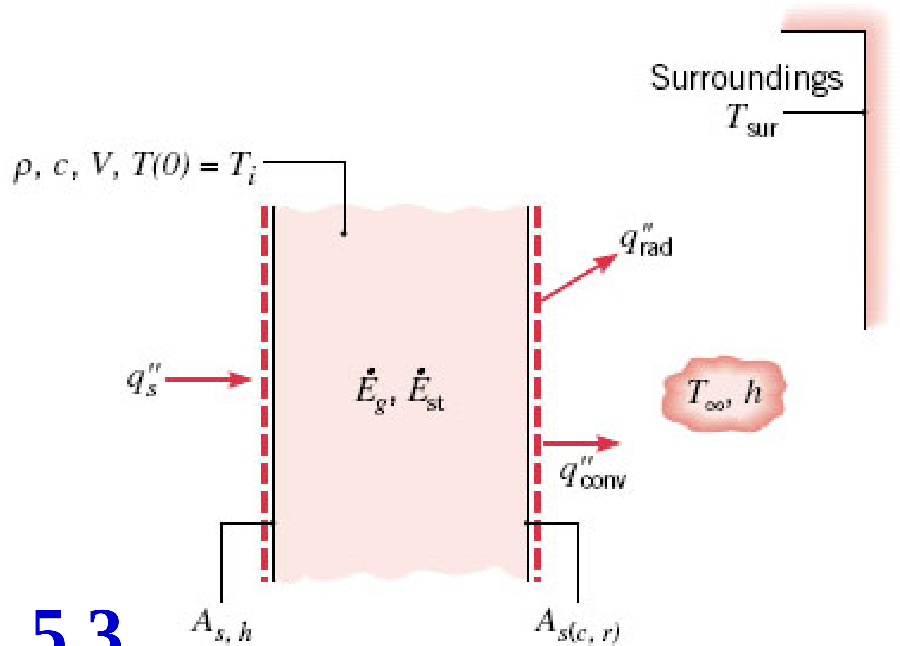


FIGURE 5.5
Control surface for general lumped capacitance analysis.

EXs 5.2, 5.3

5.4 Spatial Effects

Consider a representative 1-D heat transfer problem,

DE: $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5.26)$$

IC: $$T(x,0) = T_i \qquad (5.27)$$

BCs: $$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty] \qquad (5.28, 29)$$

Eqs. 5.26-29 imply

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h) \qquad (5.30)$$

However, after *nondimensionalization* with

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \qquad x^* \equiv \frac{x}{L} \qquad t^* \equiv \frac{\alpha t}{L^2} \equiv Fo \qquad (5.31\text{-}33)$$

$$\rightarrow \quad \frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \qquad (5.34)$$

$$\theta^*(x^*, 0) = 1 \qquad \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \qquad \left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\, \theta^*(1, t^*) \qquad (5.35\text{-}37)$$

$$\rightarrow \quad \theta^* = f(x^*, Fo, Bi) \qquad (5.38)$$

- *For a prescribed geometry, the transient temperature distribution is a universal function of x*, Fo, and Bi.* That is, the *dimensionless solution* assumes a prescribed form that does not depend on the particular value of T_i, T_∞, L, k, α, or h..
- The physical interpretation of *Fo*: *Fo* can not only viewed as the *dimensionless time*, it also provides a measure of the relative *effectiveness for a material to conduct and store energy*, as

$$(q / \dot{E}_{\mathrm{st}}) \sim (kL^2 \Delta T / L)/(\rho L^3 c \Delta T / t) = kt / \rho c L^2 = \alpha t / L^2 = Fo$$

5.5 The Plane Wall with Convection

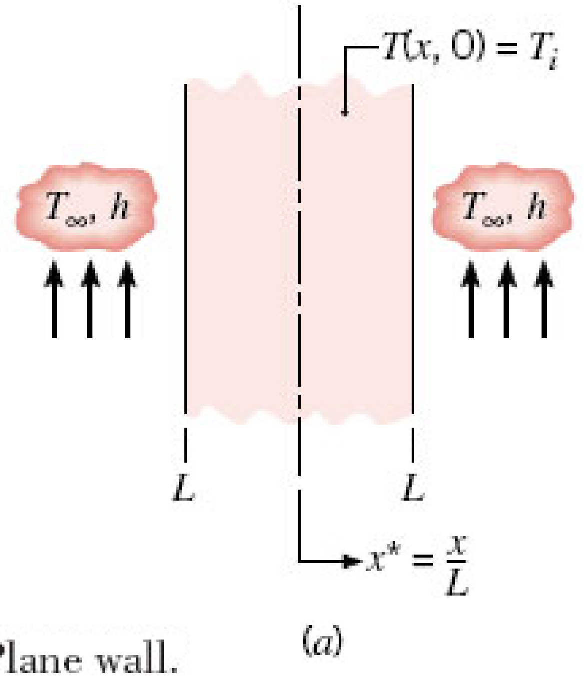


FIGURE 5.6 (*a*) Plane wall.

5.5.1 Exact Solution

The problem depicted in Fig. 5.6a has the exact solution as

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo)\cos(\zeta_n x^*) \qquad (5.39a)$$

where $$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad (5.39b)$$

and the eigenvalues ζ_n are positive roots of the transcendental eq.

$$\zeta_n \tan\zeta_n = Bi \qquad (5.39c)$$

5.5.2 Approximate Solution

For Fo > 0.2, the exact infinite series solution can be approximated by the first term of the series: ($\because \zeta_1 < \zeta_2 < \zeta_3 < \cdots$, cf. App. B.3)

$$\theta^* \approx C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) \quad (5.40a)$$

or $$\theta^* = \theta_0^* \cos(\zeta_1 x^*) \quad (5.40b)$$

where $\theta_0^* = C_1 \exp(-\zeta_1^2 Fo)$ --temperature variation at midplane $x^* = 0$.

Eq. 5.40 implies that the *time dependence* of the temperature *at any location* within the wall is *identical*.

5.5.3 Total Energy Transfer

$$Q = -\int \rho c[T(x,t) - T_i]dV; \quad Q_o = \rho c V[T_i - T_\infty]$$

$$\frac{Q}{Q_o} = \int \frac{-[T(x,t) - T_i]}{T_i - T_\infty}\frac{dV}{V} = \frac{1}{V}\int (1-\theta^*)dV \sim 1 - \frac{\sin\zeta_1}{\zeta_1}\theta_o^*$$

5.5.4 Additional Considerations

- The solution is also valid for the problem with *insulation* on one side ($x^* = 0$) and experiences convective transport on the other side ($x^* = +1$).
- The foregoing results may be used to determine the transient response of a plane wall to a sudden change in *surface* temperature ($\theta^* = \theta_i^*$, at $x^* = 1$). The process is equivalent to having an infinite convection coefficient, in which case the Biot number is infinite ($Bi = \infty$) and the fluid temperature T_∞ is replaced by the prescribed surface temperature T_s.

5.6 Radial Systems with Convection

Infinite cylinder: (Fig. 5.6b)

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*) \qquad (5.47a)$$

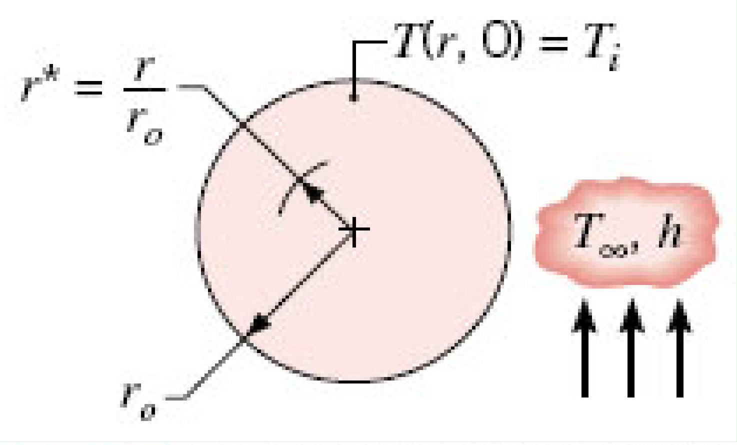


FIGURE 5.6 (b) Infinite cylinder or sphere.

where $C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)}$ (5.47b)

with eigenvalues ζ_n being the positive roots of the equation

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi \qquad (5.47c)$$

<Home Work> Solve for Eqs. 5.39 and 5.47 using the method of separation of variables.

EX 5.4

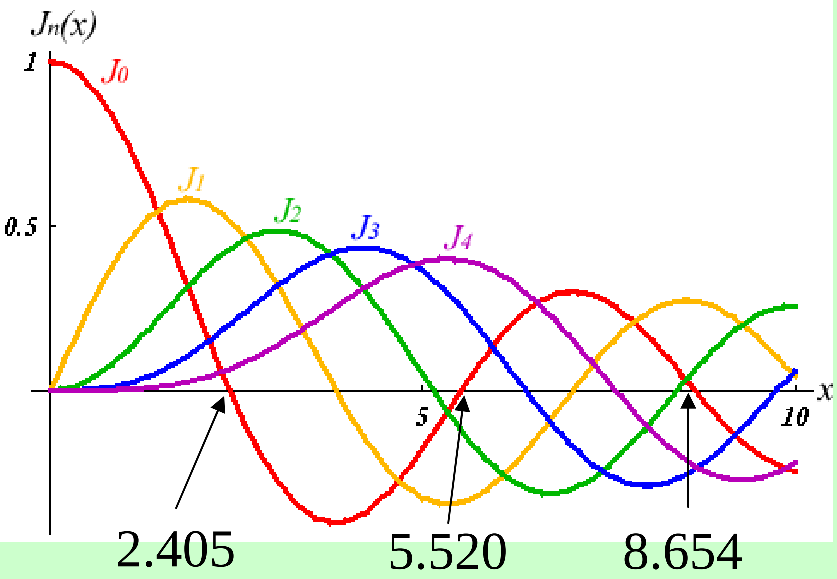
About Bessel Equation

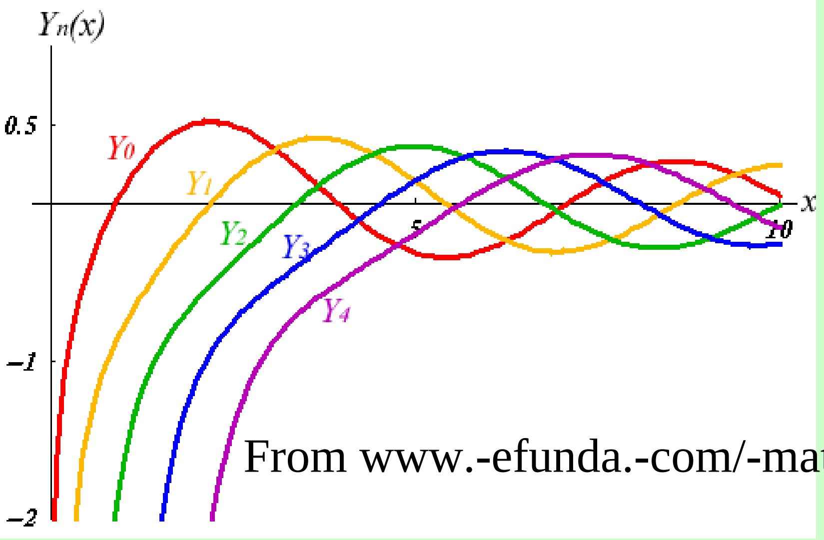
- Similar reasoning can be made for the Bessel Equation as for modified Bessel Equation, as both correspond to the cylindrical coordinate.

The Bessel Equation of order v is

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \left(1 - \frac{v^2}{r^2}\right)\theta = 0 \qquad \text{(B3)}$$

The general solution of (3) is $\theta(r) = C_1 J_\nu(r) + C_2 Y_\nu(r)$ (B3a)





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Compare $\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+m^2\theta=0$, with the general solution

$$\theta(r)=C_1J_0(mr)+C_2Y_0(mr) \qquad \text{(B4, B4a)}$$

and $\frac{d^2\theta}{dx^2}+m^2\theta=0$ (B5)

with the general solution

$$\theta(x)=C_1\cos mx+C_2\sin mx \qquad \text{(B5a)}$$

- It can be seen that $J_0(mr)$ is similar to cosmx in the oscillating behavior, but J_0 (mr) reflects damped amplitude in response to increased r. Their "periods" are also very similar except for the first half cycle. In fact, for large x

$$J_n(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

5.7 The Semi-Infinite Solid

Three cases of surface conditions: (Fig. 5.7)

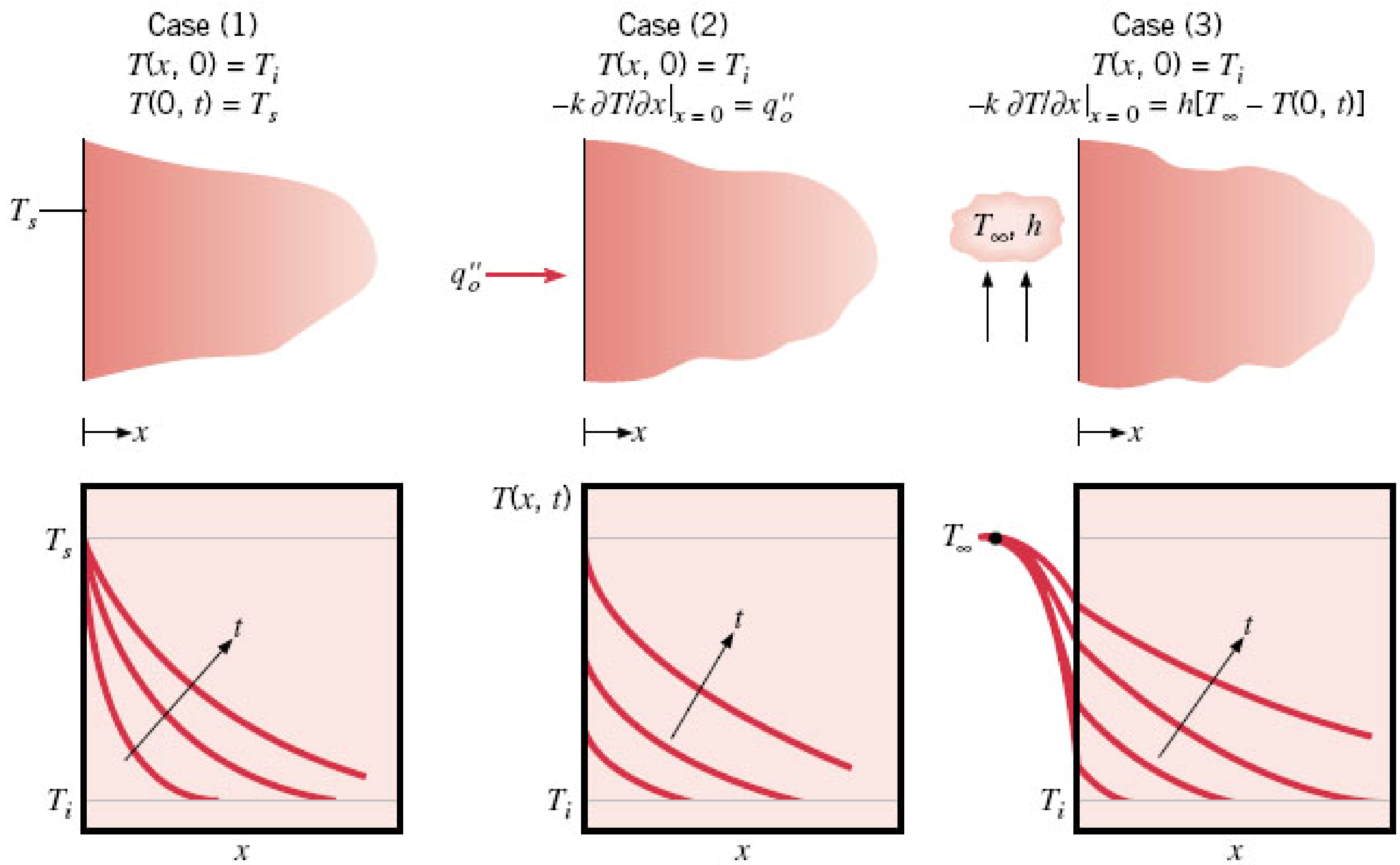


FIGURE 5.7 Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

Using the **similarity method** with *similarity variable* $\eta = x/(4\alpha t)^{1/2}$, the PDE can be transformed into an ODE. Exact solutions can thus be obtained.

- **Case 1** Constant surface temperature: $T(0,t)=T_s$

$$\frac{T(x,t)-T_s}{T_i-T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right) \quad (5.57)$$

$$q_s''(t) = \frac{k(T_s-T_i)}{\sqrt{\pi \alpha t}} \quad (5.58)$$

- **Case 2** Constant surface heat flux: $q_s'' = q_0'' \rightarrow$ Eq. 5.59
- **Case 3** Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$

  $\rightarrow$ Eq. 5.60

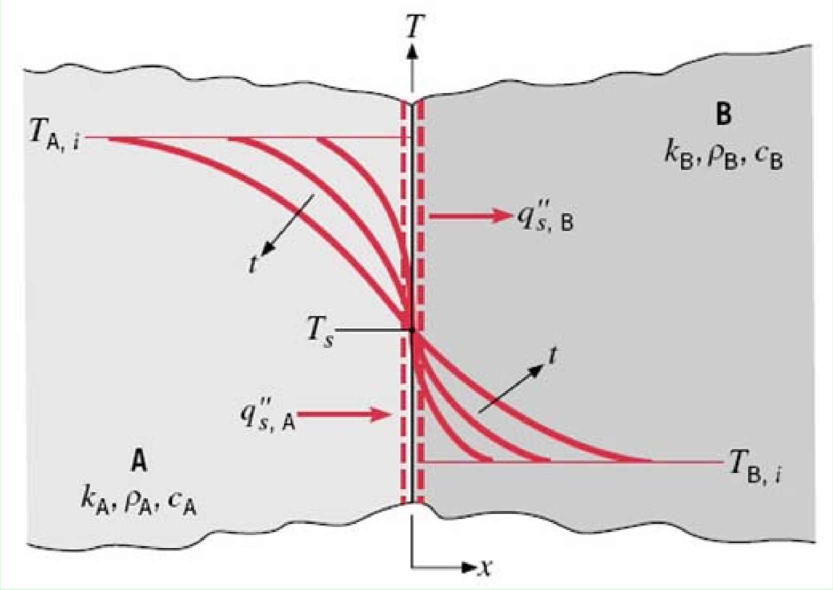
EX 5.6

A special case: interfacial contact between two semi-infinite solids at different initial temperatures

If contact resistance is negligible, there is no temperature jump at the interface. Also,

$$q''_{s,A} = q''_{s,B}$$

Since T_s does not change with time (no energy storage), from Eq 5.58



$$-\frac{k_A(T_s - T_{A,i})}{\sqrt{\pi\alpha_A t}} = \frac{k_B(T_s - T_{B,i})}{\sqrt{\pi\alpha_B t}} \quad (5.62)$$

$$\rightarrow T_s = \frac{\sqrt{(k\rho c)_A}T_{A,i} + \sqrt{(k\rho c)_B}T_{B,i}}{\sqrt{(k\rho c)_A} + \sqrt{(k\rho c)_B}} \quad (5.63)$$

With T_s determined, the temperature profile can be described by Eq. 5.57.