Chapter 3

Steady-State,
One-Dimensional Conduction

3.1 The Plane Wall

3.1.1 Temperature Distribution

For one-dimensional, steady-state conduction in a plane wall with no heat generation, the differential equation (2.13) reduces to

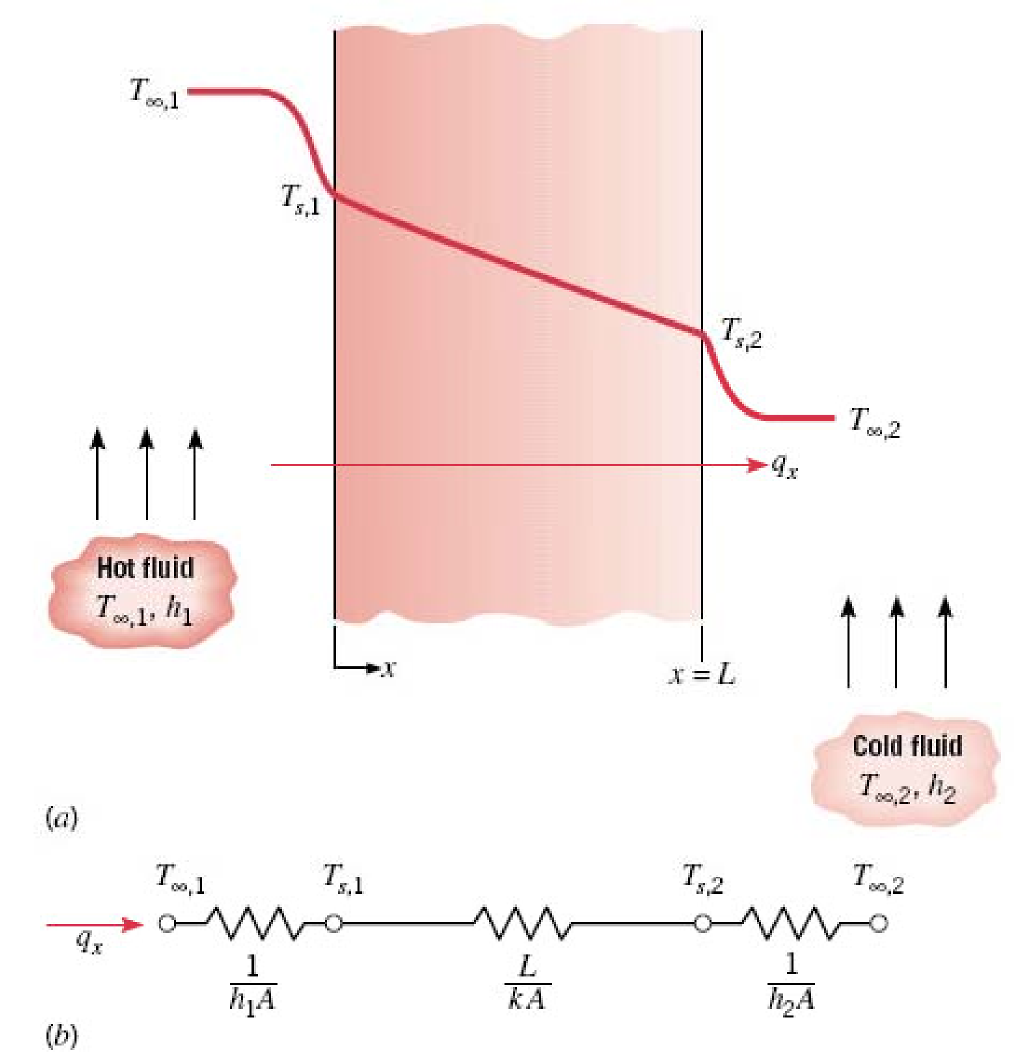
$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \qquad (3.1)$$

and the heat flux is a constant, independent of x. With further assumption of constant , we have the general linear solution

$$T(x)=C_1x+C_2 \qquad (3.2)$$

The **heat flux** is

$$q''_x=\frac{q_x}{A}=\frac{k}{L}(T_{s,1}-T_{s,2}) \qquad (3.5)$$



3.1.2 Thermal Resistance

With the analogy between the diffusion of heat and electrical charge, the **thermal resistance** for conduction is

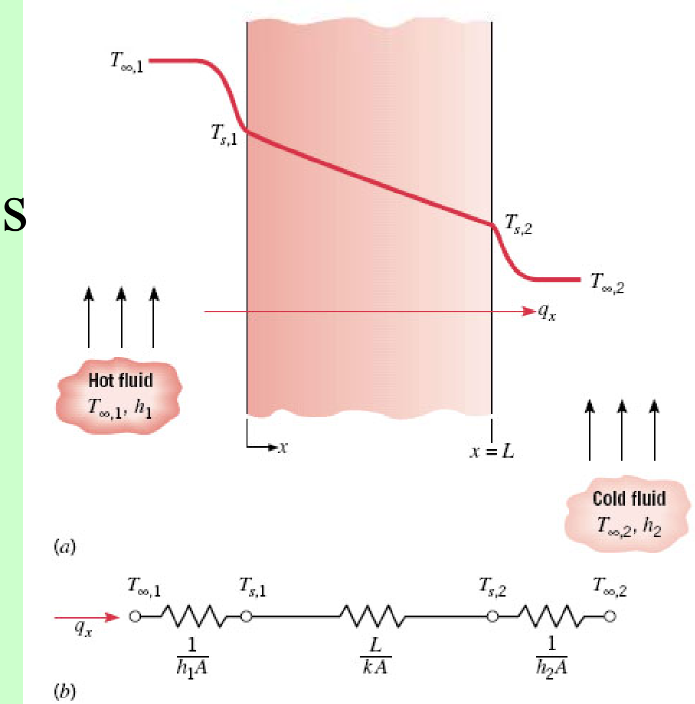
$$R_{t,\text{cond}} = \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA} \qquad (3.6)$$

Similar for convection with Newton's law of cooling

$$R_{t,\text{conv}} = \frac{T_s - T_\infty}{q} = \frac{1}{hA} \qquad (3.9)$$

In Fig. 3.1, the **total thermal resistance**, R_{tot}, is

$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad (3.12)$$



3.1.3 The Composite Wall (Fig. 3.2; Fig. 3.3)

$$R_{\text{tot}} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA} \tag{3.19}$$

where U is the **overall heat transfer coefficient**, defined by analogy to Newton's law of cooling as

$$q_x \equiv UA\Delta T$$

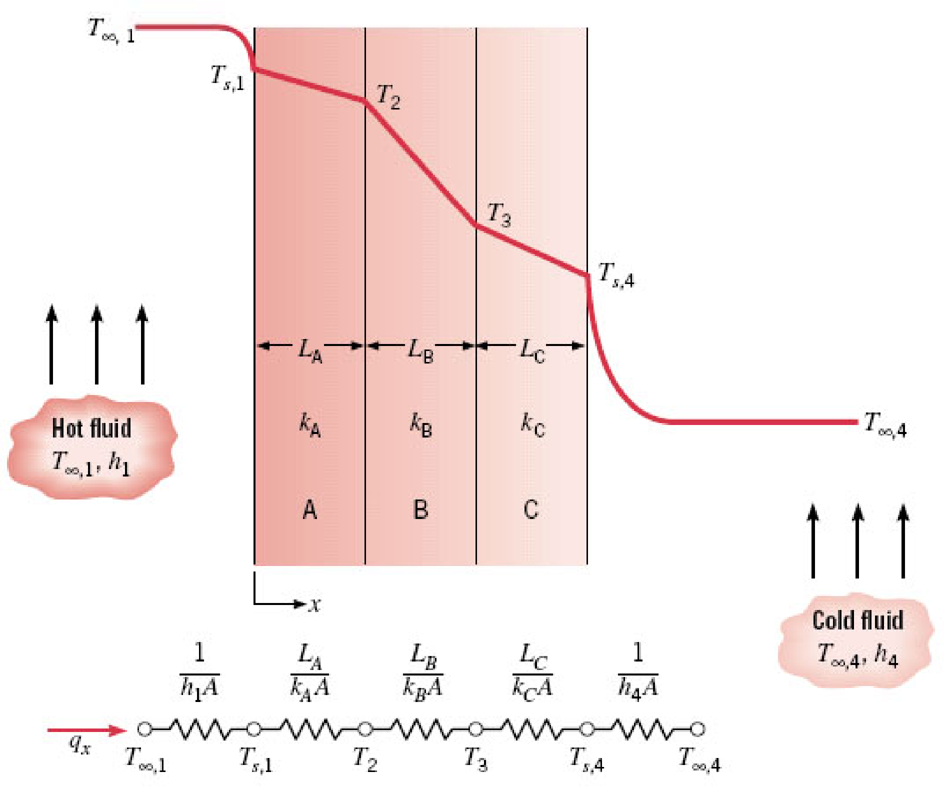


FIGURE 3.2 Equivalent thermal circuit for a series composite wall.

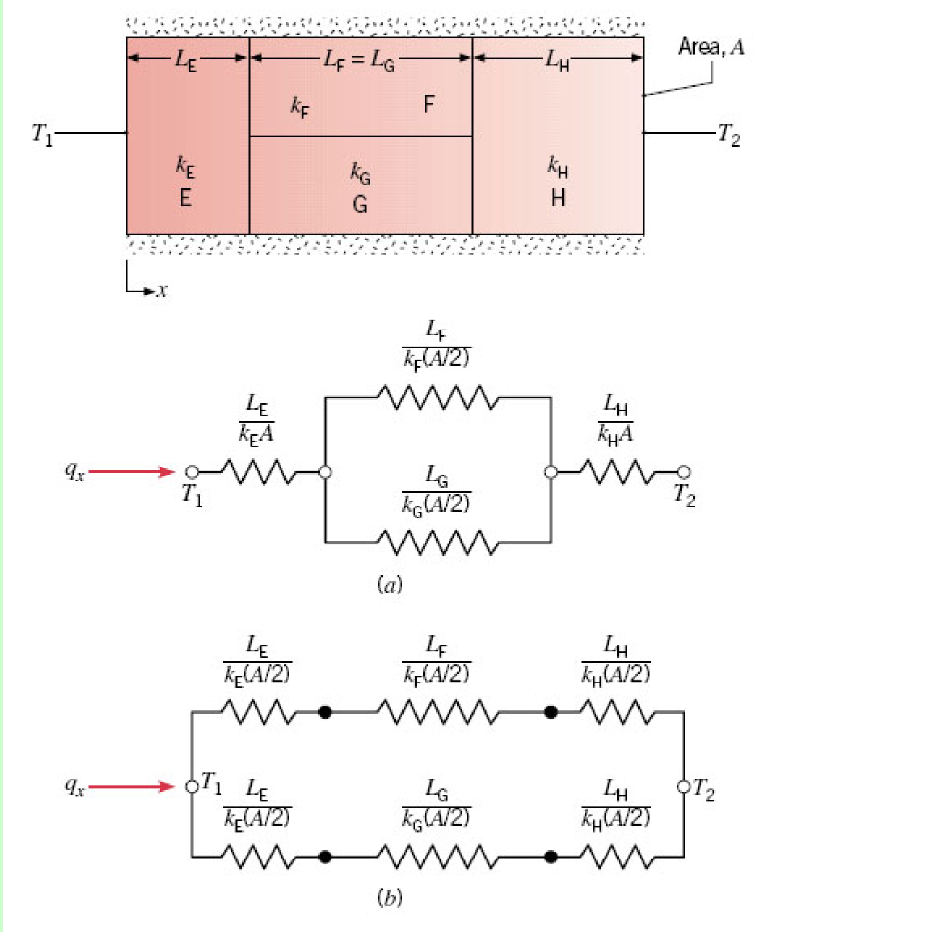


FIGURE 3.3 Equivalent thermal circuits for a series–parallel composite wall.

3.1.4 Contact Resistance

--significant temperature drop exists across an interface due to the gaps between the contact area

- **Contact resistance** can be reduced by increasing the joint pressure, reducing the roughness of the mating surfaces, applying a metal coating, inserting the interface with soft metal foil, or filling fluid of large thermal conductivity, etc. See Tables 3.1 and 3.2.

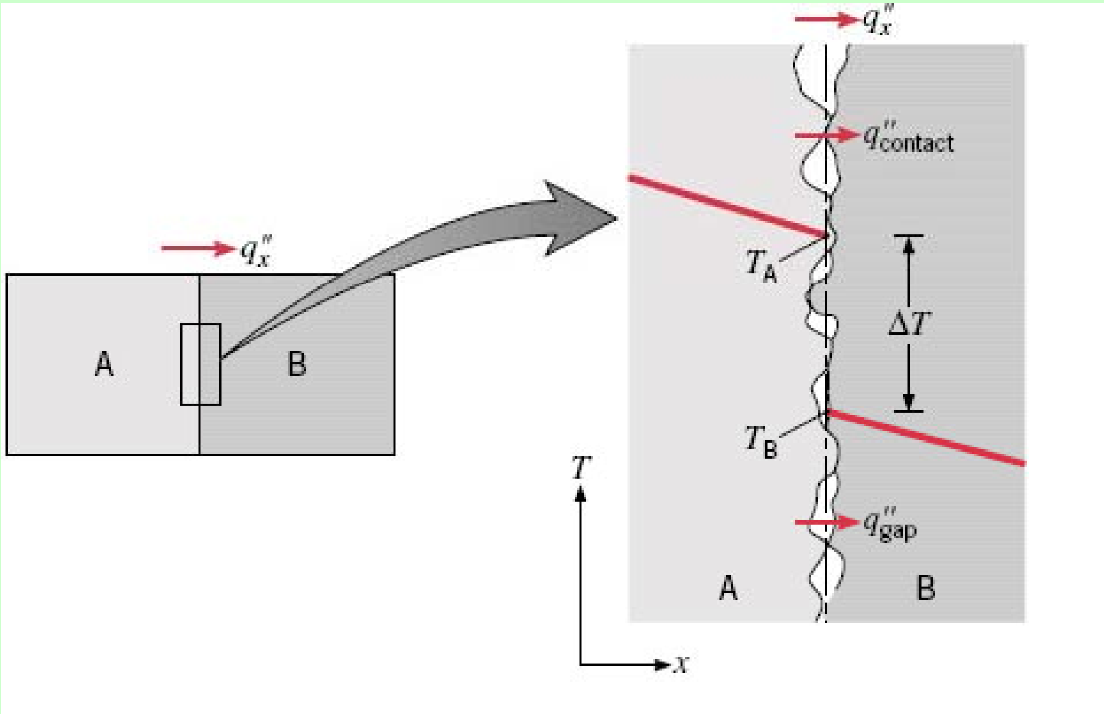


FIGURE 3.4 Temperature drop due to thermal contact resistance.

TABLE 3.1 Thermal contact resistance for (*a*) metallic interfaces under vacuum conditions and (*b*) aluminum interface (10-μm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

Thermal Resistance, $R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)

(*a*) Vacuum Interface			(*b*) Interfacial Fluid	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

TABLE 3.2 Thermal resistance of representative solid/solid interfaces

Interface	$R''_{t,c} \times 10^4$ ($m^2 \cdot K/W$)	Source
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6	[2]
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04	[1, 3]
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1	[4]
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04	[1, 3]
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9	[5]
Brass/brass with 15-μm tin solder	0.025–0.14	[6]

EX 3.1, 3.2, 3.3

3.3 Radial Systems

3.3.1 The Cylinder

For steady-state conditions with no heat generation, Eq. 2.20 is reduced to

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \qquad (3.23)$$

The conduction heat transfer rate q_r (<u>not the heat flux</u> q_r'') is a constant in the radial direction

$$q_r = -kA\frac{dT}{dr} = -k(2\pi rL)\frac{dT}{dr} \qquad (3.24)$$

The general solution of (3.23) is

$$T(r) = C_1 \ln r + C_2 \qquad (3.25)$$

An example is shown in Fig. 3.6.

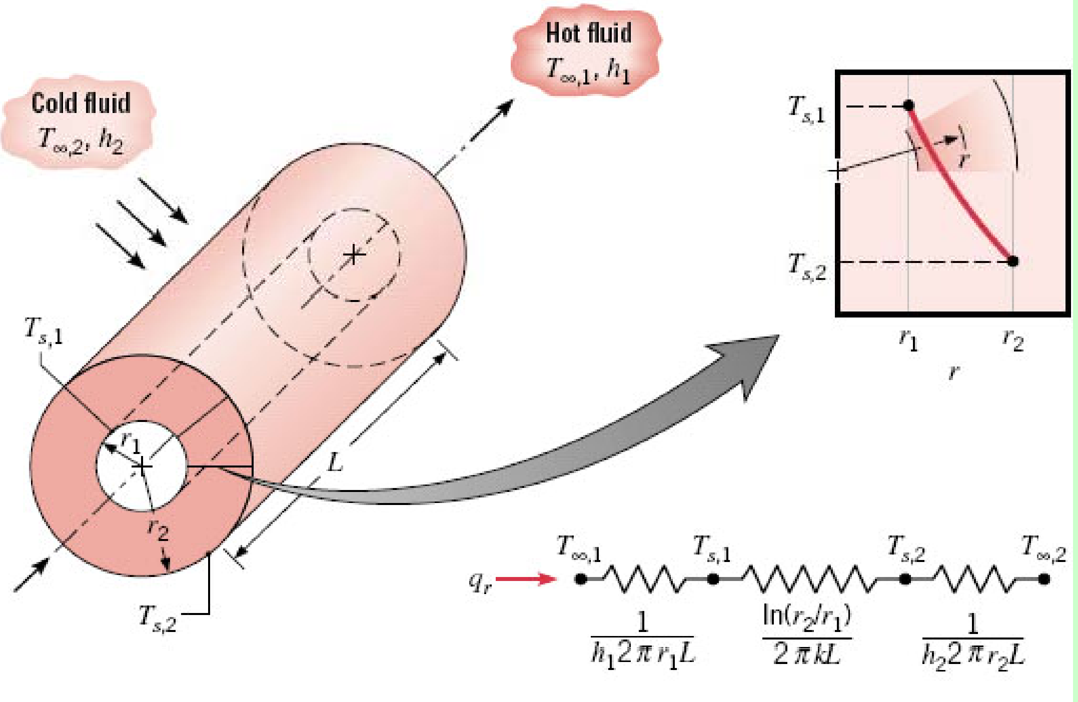


FIGURE 3.6 Hollow cylinder with convective surface conditions.

$$T(r)=\frac{T_{s,1}-T_{s,2}}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right)+T_{s,2} \qquad (3.26)$$

$$q_r=\frac{2\pi Lk(T_{s,1}-T_{s,2})}{\ln(r_2/r_1)} \qquad (3.27)$$

$$R_{t,\text{cond}}=\frac{\ln(r_2/r_1)}{2\pi Lk} \qquad (3.28)$$

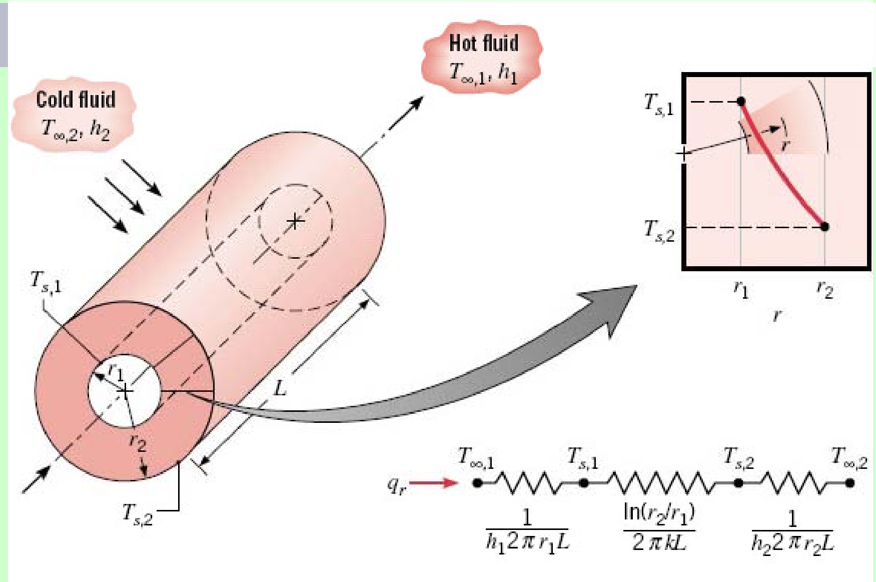


FIGURE 3.6 Hollow cylinder with convective surface conditions.

The temperature distribution for a composite cylindrical wall is shown in Fig. 3.7.

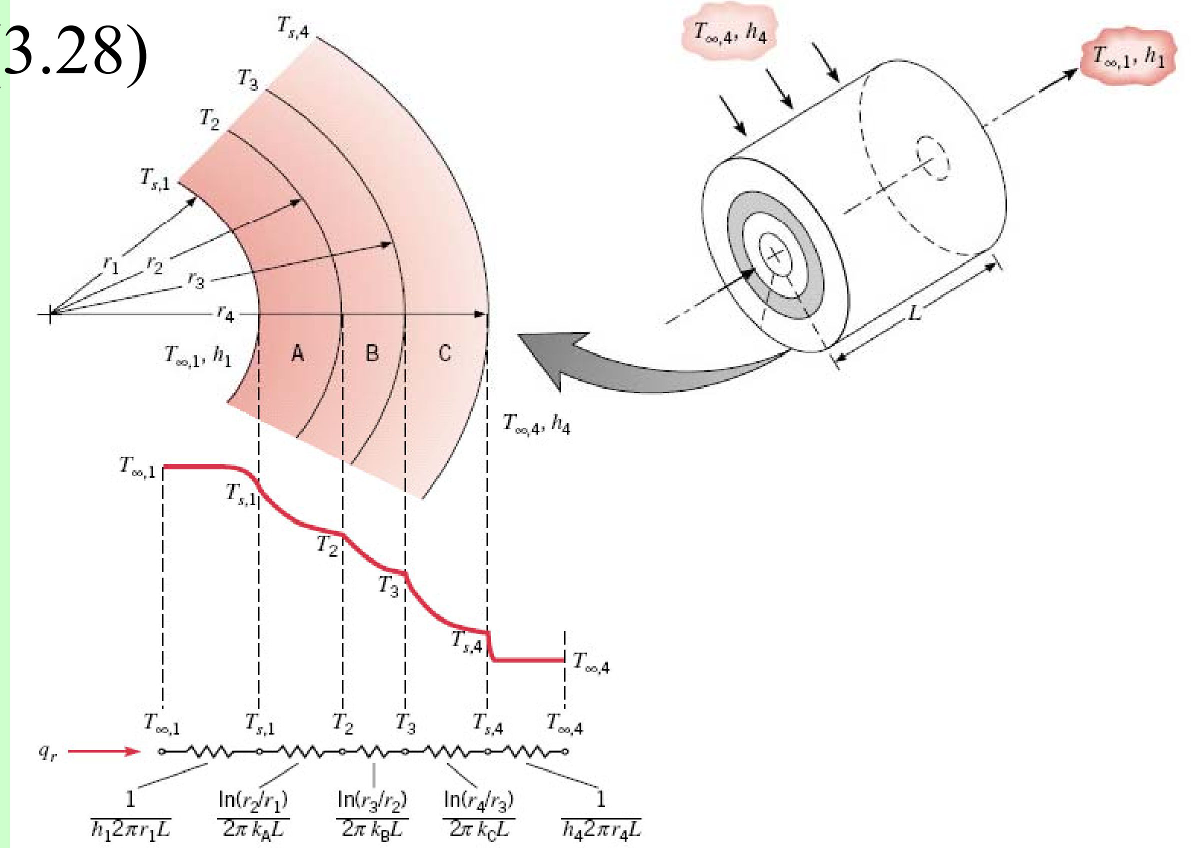


FIGURE 3.7 Temperature distribution for a composite cylindrical wall.

EX 3.5

TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr} = k/h$ for the cylinder and $r_{cr} = 2k/h$ for the sphere.

3.5 Conduction with Thermal Energy Generation

3.5.1 The Plane Wall

For steady-state conditions with constant *k* and *uniform* energy generation per unit volume, Eq. 2.16 becomes

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0 \qquad (3.39)$$

The general solution is

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Some examples are shown in Fig. 3.9. With heat generation the heat flux is *no longer independent of x*.

EX 3.7

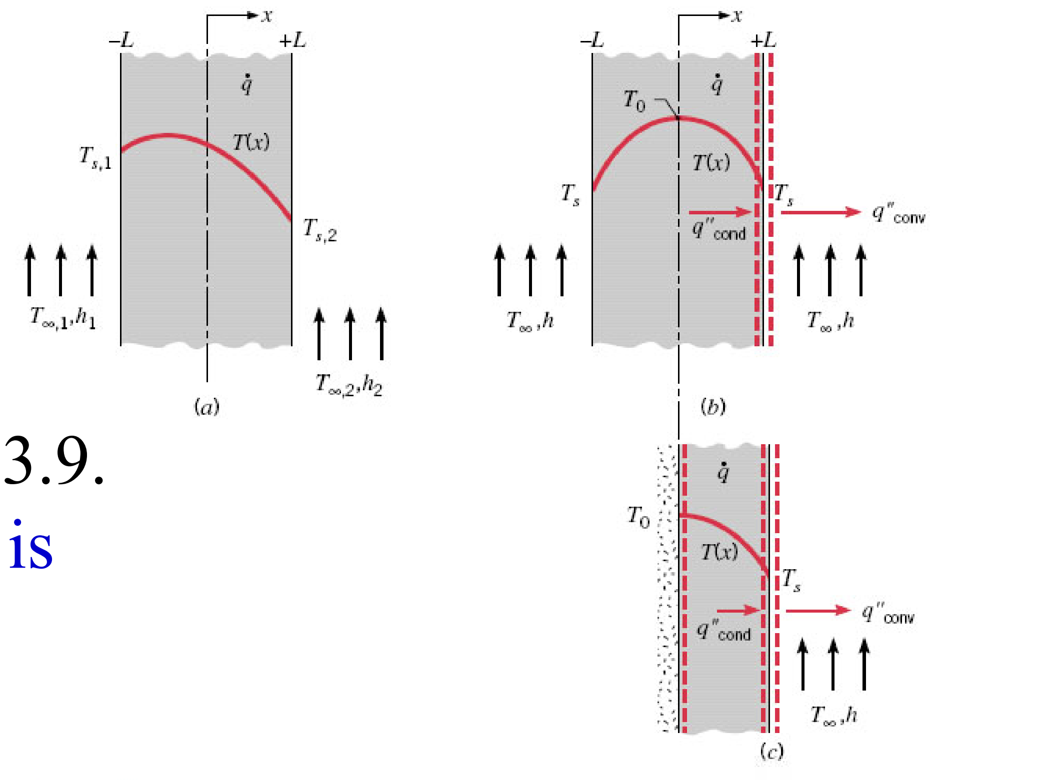


FIGURE 3.9 Conduction in a plane wall with uniform heat generation. (*a*) Asymmetrical boundary conditions. (*b*) Symmetrical boundary conditions. (*c*) Adiabatic surface at midplane.

3.6 Heat Transfer from Extended Surfaces

Use of fins to enhance heat transfer from a wall: Figs. 3.12-3.14

3.6.1 A General Conduction Analysis (Fig. 3.15)

Through energy balance, we obtain

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)-\frac{h}{k}\frac{dA_s}{dx}(T-T_\infty)=0$$

or $$\frac{d^2T}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx}-\left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T-T_\infty)=0 \qquad (3.61)$$

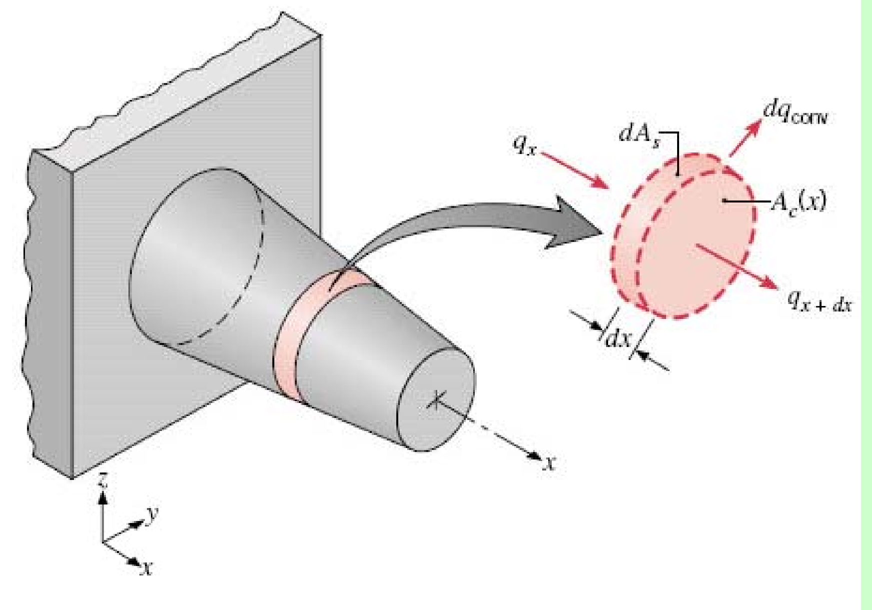


FIGURE 3.15 Energy balance for an extended surface.

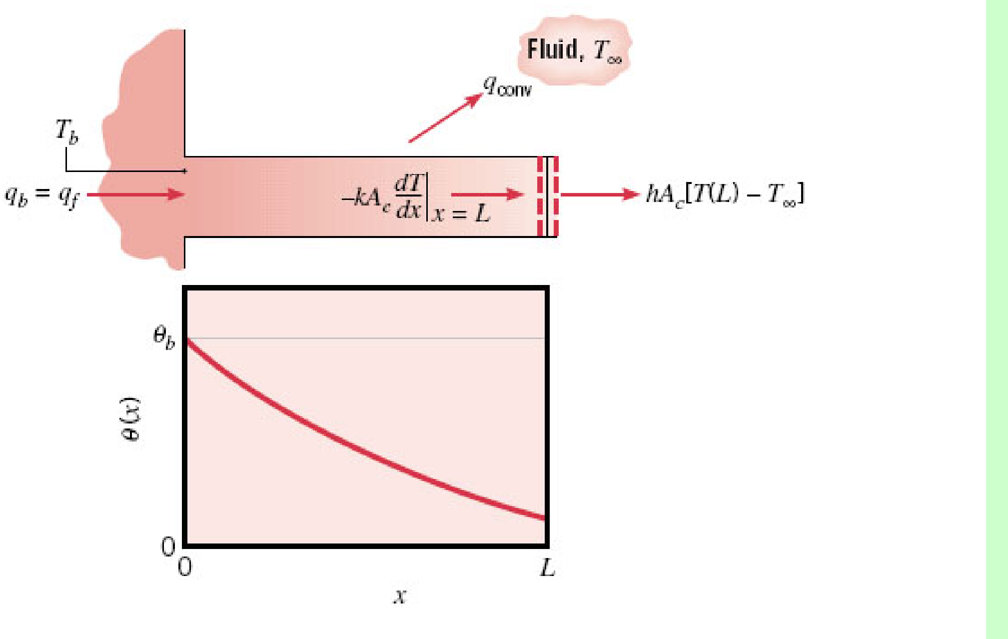


FIGURE 3.17 Conduction and convection in a fin of uniform cross section.

$\theta(0) = T_b - T_\infty \equiv \theta_b$

3.6.2 Fins of *Uniform* Cross-Sectional Area

With $dA_c/dx = 0$ and $dA_s/dx = P$, Eq. 3.61 reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad (3.62)$$

Defining an *excess temperature* θ as $\theta(x) \equiv T(x) - T_\infty$ (3.63)

$$\Rightarrow \frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \text{where } m^2 \equiv \frac{hP}{kA_c} \qquad (3.64,65)$$

The general solution of (3.64) is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \qquad (3.66)$$

Two boundary conditions are needed.

- One boundary condition can be specified at the base of the fin ($x = 0$) as

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

- The second condition may correspond to one of the four physical situations shown in Table 3.4. The solution procedure to obtain temperature distribution θ/θ_b and fin heat transfer rate q_f is discussed in the textbook.

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.70)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.72)
B	Adiabatic $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	(3.75)	$M \tanh mL$	(3.76)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.78)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.79)	M	(3.80)

$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\theta_b$

EX 3.9

Proper Length of a Fin

- Heat transfer ratio between a fin of finite length L and a fin of infinite length $= \dfrac{q_{f,L}}{q_{f,L\to\infty}} \cong \dfrac{(3.76)}{(3.80)} = \tanh mL$

  $\rightarrow$fin length $L < 2.65/m$

In practice, fin length is usually constrained by space and weight.

3.6.3 Fin Performance

•Fin effectiveness ε_f: ratio of the fin heat transfer rate to that without the fin

$$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b} \qquad (3.81)$$

Use of fins may rarely be justified unless $\varepsilon_f \gamma 2$.

$$\varepsilon_f \approx \left(\frac{kP}{hA_c}\right)^{1/2} \qquad (3.82)$$

mL	$\tanh mL$
0.1	0.100
0.5	0.462
1.0	0.762
2.0	0.964
2.5	0.987
2.65	0.990
3.0	0.995
5.0	1.000

Consider an infinite fin, assuming h unaltered by the presence of fin,

$\rightarrow$ $\varepsilon_f \uparrow$as k, $P/A_c\uparrow$(high k material; thin fins)

$h \downarrow$ (gas medium and/or natural convection)

- Fin resistance $R_{t,f}$: $$R_{t,f} = \frac{\theta_b}{q_f} \quad (3.83)$$
- Fin efficiency η_f: ratio of q_f and the maximum rate when the entire fin surface were at the base temperature

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\theta_b} \quad (3.86)$$

For a straight fin of uniform cross section and an adiabatic tip, Eqs. 3.76 and 3.86 yield

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL} \quad (3.87)$$

This expression can be applied to a fin with convective tip, if a corrected fin length L_c (eg., $L_c = L + t/2$) for a rectangular fin) is used. The fin efficiency is shown in Fig. 3.18.

- The fin resistance can be written in terms of η_f.

$$R_{t,f} = \frac{1}{hA_f\eta_f} \quad (3.92)$$

The η_f and A_f for some examples are shown in Table 3.5.

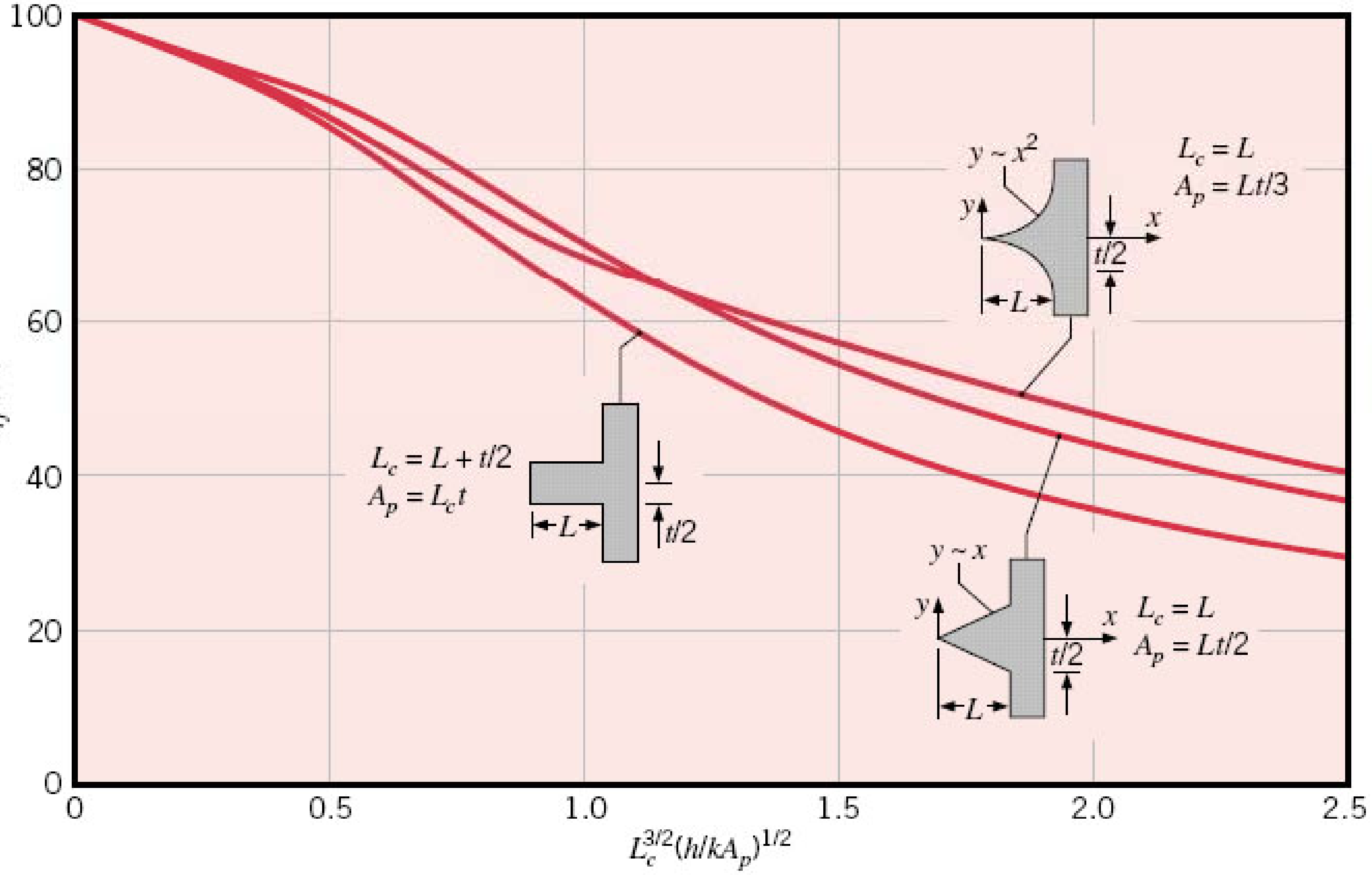


FIGURE 3.18 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

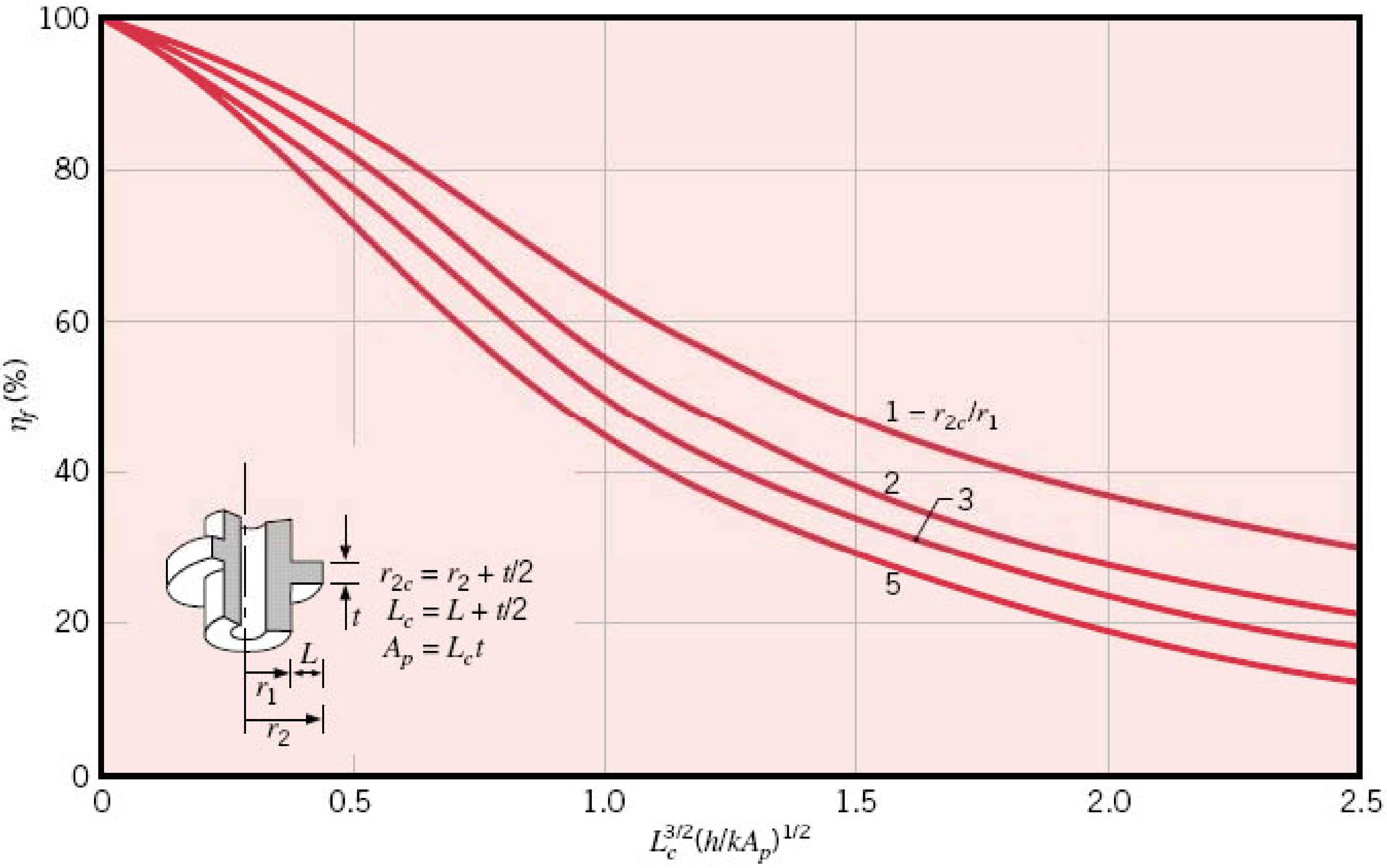


FIGURE 3.19 Efficiency of annular fins of rectangular profile.

3.6.4 Fins of *Nonuniform* Cross-Sectional Area

For an annular fin with uniform fin thickness,

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}-\frac{2h}{kt}(T-T_\infty)=0$$

with $m^2 = 2h/kt$, $\theta = T\text{-}T_\infty$,

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-m^2\theta=0$$ -- modified Bessel eq. of order zero

The general solution is of the form

$$\theta(r)=C_1I_0(mr)+C_2K_0(mr)$$

where I_0 and K_0 are modified, zero-order Bessel functions of the first and second kinds, respectively.

<Home Work> Solve for the expression of θ/θ_b, given the on p.151, for the case with a given base temperature and an adiabatic tip.

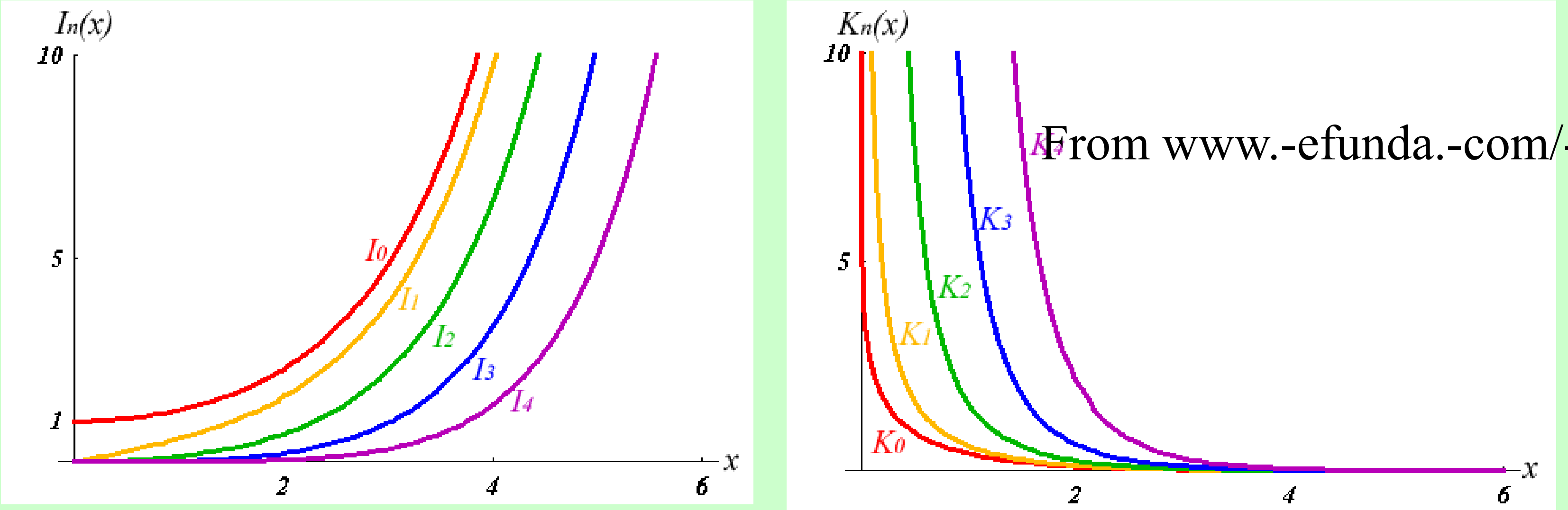
The fin efficiencies of some examples with nonuniform cross-sectional area are shown in Figs. 3.18 and 3.19.

About Modified Bessel Equation

Modified Bessel Equation of order v

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-(1+\frac{v^2}{r^2})\theta=0 \qquad \text{(B1)}$$

The general solution of (1) is $\theta(r)=C_1 I_v(r)+C_2 K_v(r)$



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Consider $\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-m^2\theta=0$ (B1')

Let $r'=mr$, it becomes $\frac{d^2\theta}{dr'^2}+\frac{1}{r'}\frac{d\theta}{dr'}-\theta=0$

which is a modified Bessel equation of order zero, with the general solution of the form $\theta(r)=C_1 I_0(mr)+C_2 K_0(mr)$ (B1a)

Compare $\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}-m^2\theta=0$, having the general solution

$$\theta(r)=C_1 I_0(mr)+C_2 K_0(mr) \qquad \text{(B1a)}$$

with $$\frac{d^2\theta}{dx^2}-m^2\theta=0 \qquad \text{(B2)}$$

which has the general solution

$$\theta(x)=C_1 e^{mx}+C_2 e^{-mx} \qquad \text{(B2a)}$$

- Since the modified Bessel equation corresponds to the cylindrical coordinate, while (2) corresponds to the Cartesian coordinate, their solutions are similar in trend but (1a) reflects the effect of increasing surface area $2\pi r$ with increasing r.

3.6.5 Overall Surface Efficiency

--characterizes an *array* of fins (Fig. 3.20) and the base surface to which they are attached

$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{hA_t\theta_b} \qquad (3.98)$$

with $A_t = NA_f + A_b$ and $q_t = N\eta_f hA_f\theta_b + hA_b\theta_b$, we obtain

$$\eta_o = 1 - \frac{NA_f}{A_t}(1 - \eta_f) \qquad (3.102)$$

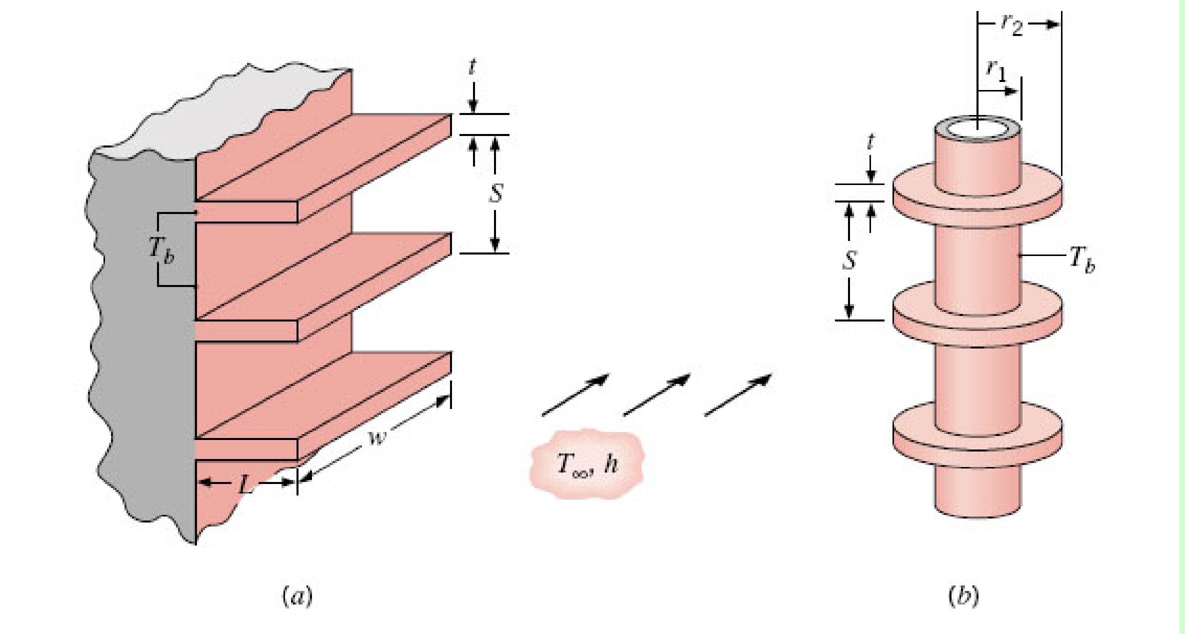

FIGURE 3.20 Representative fin arrays. (*a*) Rectangular fins. (*b*) Annular fins.

If the fins are attached to the base, rather than an integral part of the wall, contact resistance should be included, as in Fig. 3.21 and Eqs. 3.104-105.

$$R_{t,o(c)} = \frac{\theta_b}{q_t} = \frac{1}{\eta_{o(c)} h A_t} \quad (3.104)$$

$$\eta_{o(c)} = 1 - \frac{N A_f}{A_t}\left(1 - \frac{\eta_f}{C_1}\right) \quad (3.105)$$

where

$$C_1 = 1 + \eta_f h A_f (R''_{t,c} / A_{c,b})$$

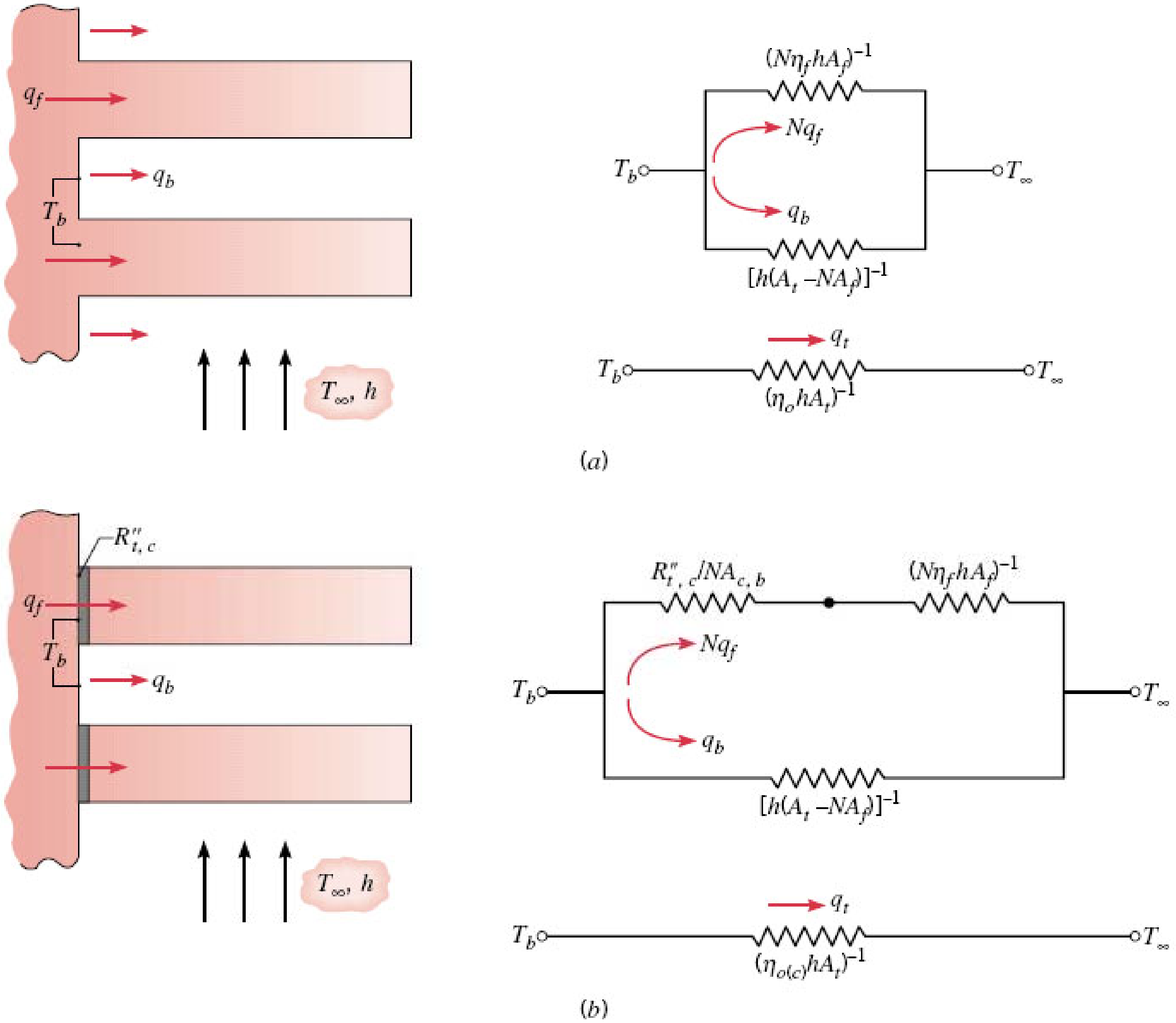

FIGURE 3.21 Fin array and thermal circuit. (a) Fins that are integral with the base. (b) Fins that are attached to the base.

EXs 3.10, 3.11