Chapter 2

Fundamental Concepts of Conduction

2.1 The Conduction Rate Equation

Fourier's law: from observation

$$\vec{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial x} + \boldsymbol{j}\frac{\partial T}{\partial y} + \boldsymbol{k}\frac{\partial T}{\partial z}\right) \qquad (2.3)$$

$$= \boldsymbol{i}q''_x + \boldsymbol{j}q''_y + \boldsymbol{k}q''_z \qquad (2.5)$$

where $q''_x = -k\frac{\partial T}{\partial x} \quad q''_y = -k\frac{\partial T}{\partial y} \quad q''_z = -k\frac{\partial T}{\partial z}$ (2.6)

or $q''_n = -k\frac{\partial T}{\partial n}$ (2.4)

where q''_n is the heat flux in a direction $\vec{n}$, which is normal to an *isotherm* (Fig. 2.3).

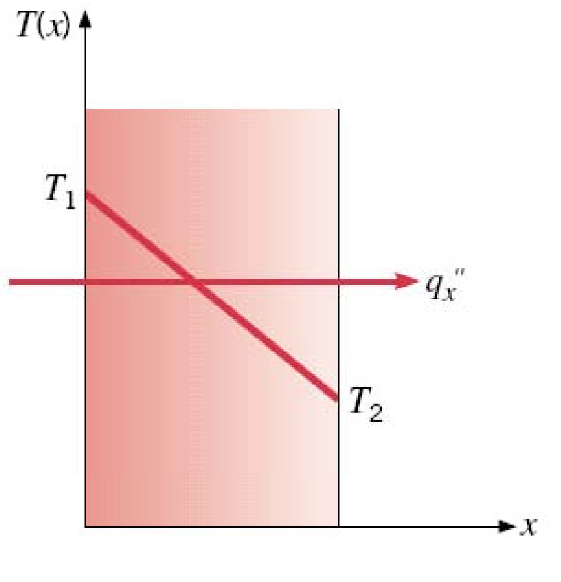


FIGURE 2.2

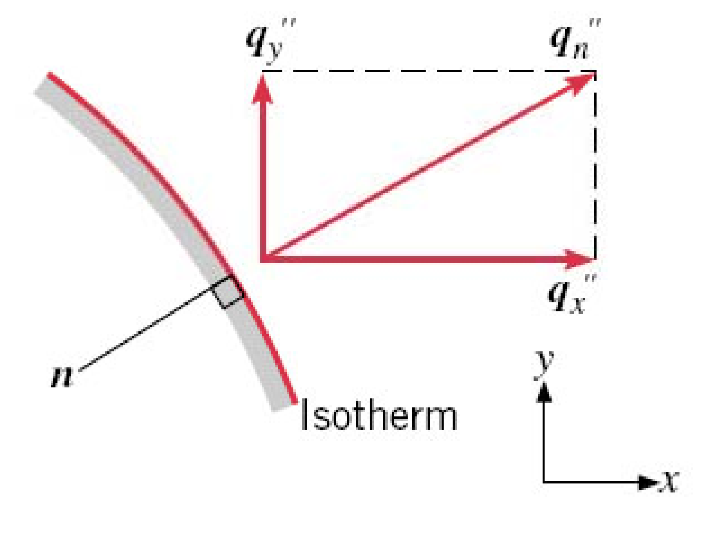
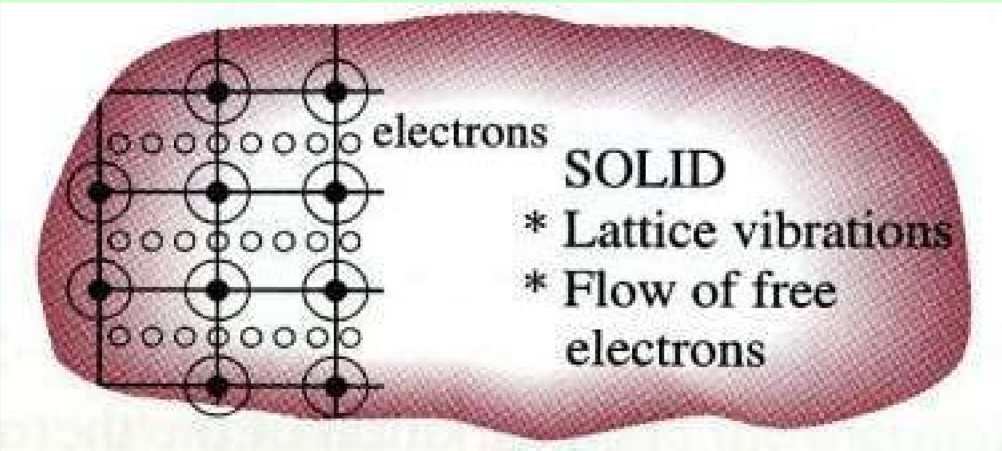


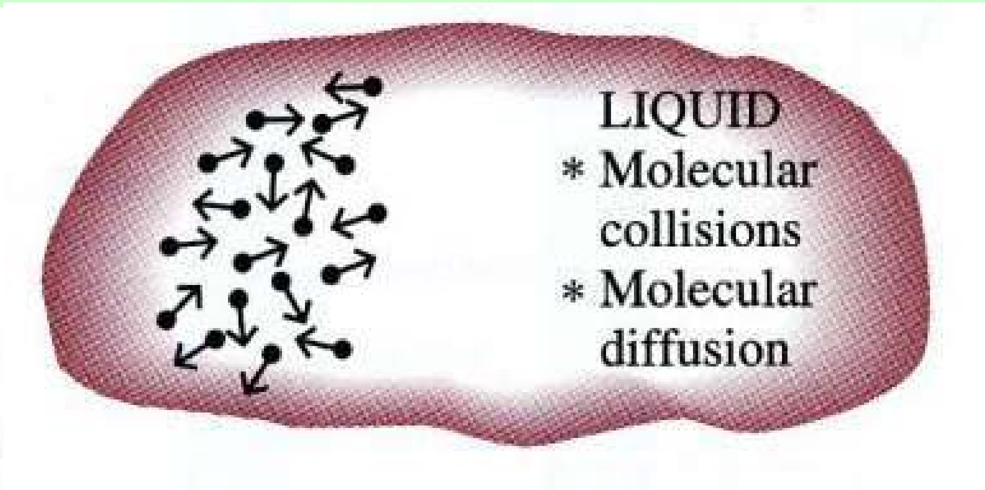
FIGURE 2.3

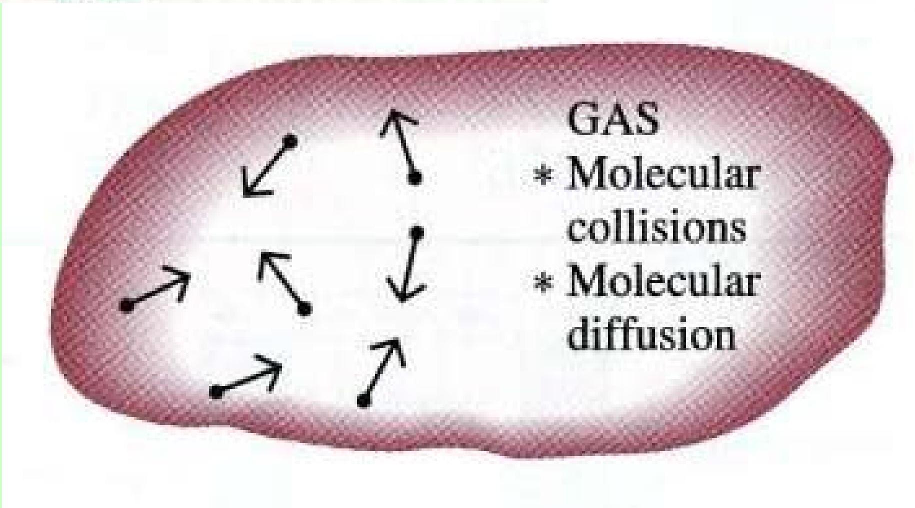
2.2 The Thermal Properties of Matter

2.2.1 Thermal Conductivity

$$k \equiv -\frac{q''_x}{(\partial T / \partial x)}$$







From Cengel, Heat Transfer, 2003.

The Solid State: due to migration of free electrons (k_e) and lattice vibrational waves. When viewed as a particle-like phenomenon, the lattice vibration quanta are termed *phonons*. From kinetic theory,

$$k = \frac{1}{3} C \,\overline{c}\, \lambda_{\mathrm{mfp}} \tag{2.7}$$

- For nonconducting solids,

  $C \equiv C_{\mathrm{ph}}$: the phonon specific heat per unit volume

  $\overline{c}$: the average speed of sound

  $\lambda_{\mathrm{mfp}} \equiv \lambda_{\mathrm{ph}}$: the phonon mean free path

- For conducting materials such as metals,

  $C \equiv C_e$: the electron specific heat per unit volume

  $\overline{c}$: the mean electron velocity

  $\lambda_{\mathrm{mfp}} \equiv \lambda_e$: the electron mean free path

- In general, $k = k_e + k_{\mathrm{ph}}$ (2.8)

Generally, $k_{metal} > k_{crystalline} > k_{amorphous}$ (exceptions: diamond, silicon carbide)

$k_{pure\ metal} > k_{alloy}$ (At 300K, 2300W/mK, 490W/mK)

(Figs. 2.4 & 2.5)

Source: Y.A. Cengel

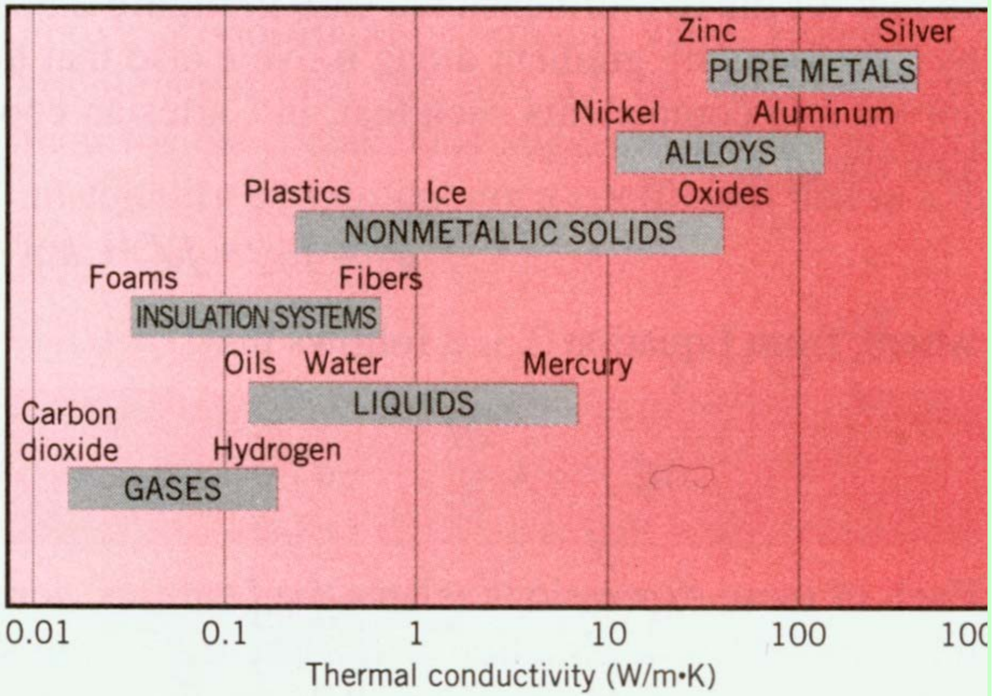
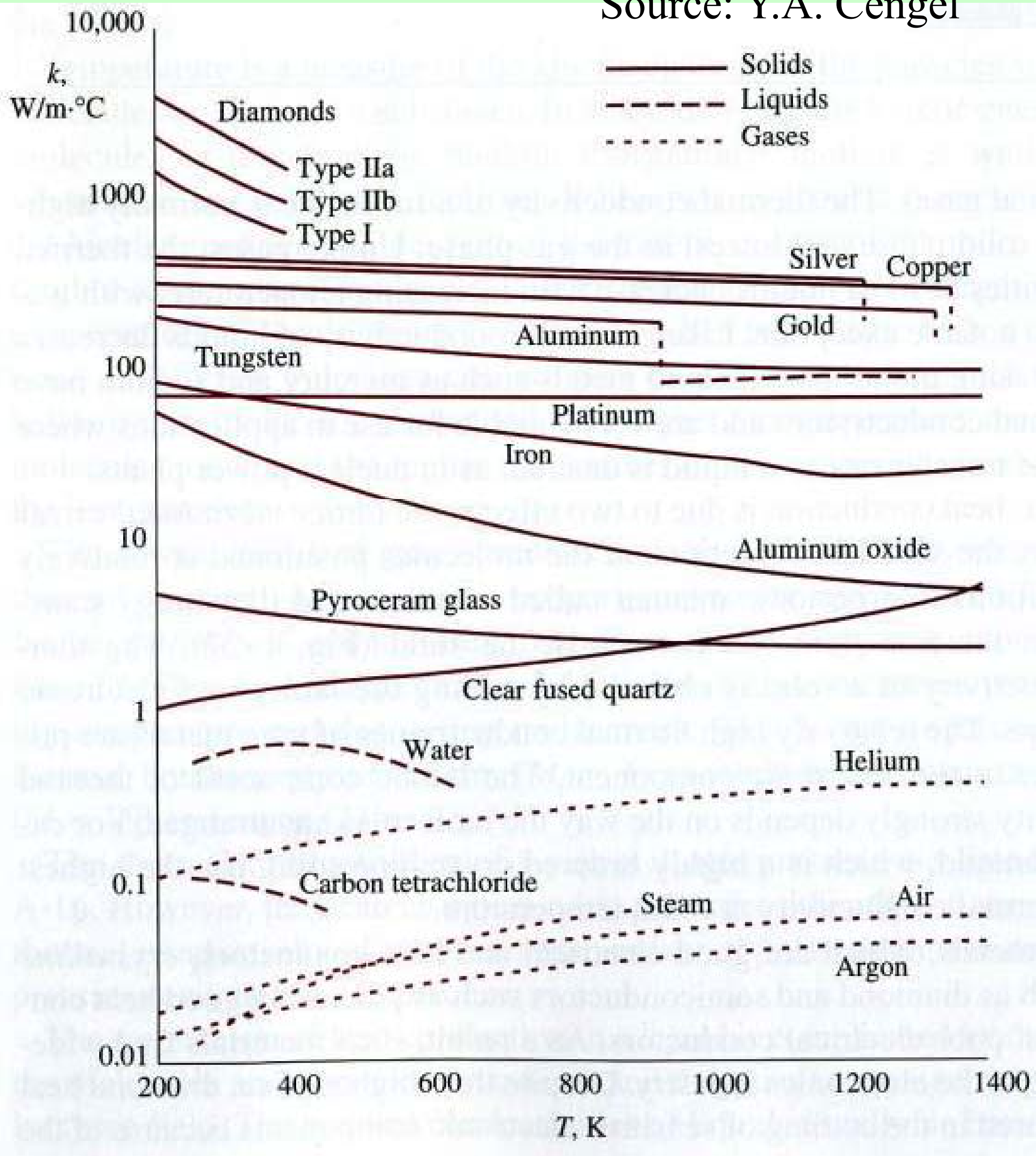


FIGURE 2.4 Range of thermal conductivity for various states of matter at normal temperatures and pressure.



The Solid State: Micro- and Nanoscale Effects

- In many areas of technology, such as microelectronics, the material dimensions can be on the order of micrometers or nanometers, in which care must be taken to account for the possible modifications of k that can occur as the physical dimensions become small.
- Cross sections of *films* of the same material having thicknesses L_1 and L_2 are shown in Fig. 2.6. Note that the physical boundaries of the film act to *scatter* the energy carriers and *redirect* their propagation. We find that $k_y < k_x < k$, where k is the bulk thermal conductivity.

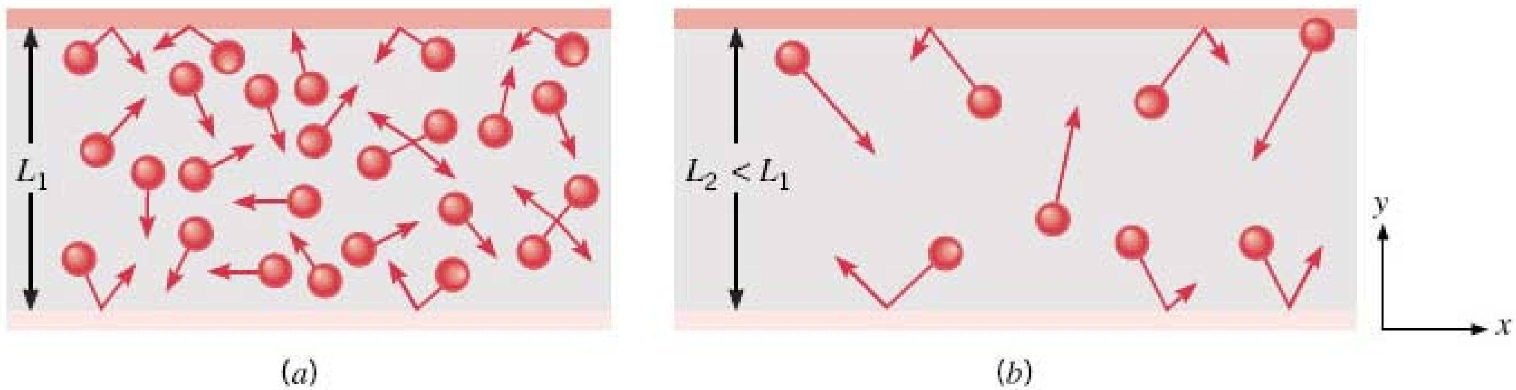


FIGURE 2.6 Electron or phonon trajectories in (a) a relatively thick film and (b) a relatively thin film with boundary effects.

- For $L/\lambda_{\mathrm{mfp}} \geq 1$, predicted values of k_x and k_y may be estimated to within 20% from

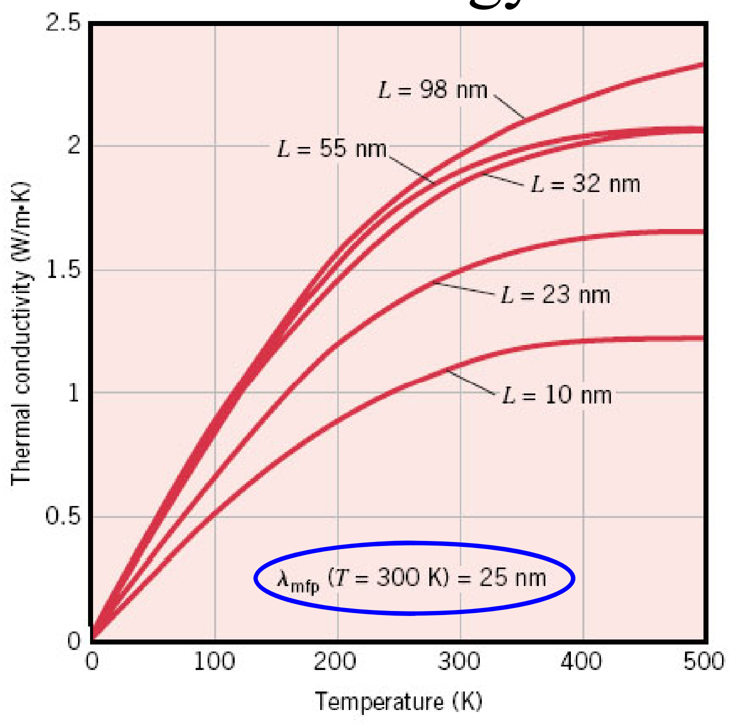
$$k_x / k = 1 - 2\lambda_{\mathrm{mfp}} / (3\pi L) \qquad (2.9)$$

$$k_y / k = 1 - \lambda_{\mathrm{mfp}} / (3L)$$

- Eq. 2.9 reveals that the values of k_x and k_y are within 5% of the bulk k if $L/\lambda_{\mathrm{mfp}} > 7$ (for k_y) and $L/\lambda_{\mathrm{mfp}} > 4.5$ (for k_x). No general guidelines exist for $L/\lambda_{\mathrm{mfp}} < 1$.
- Chemical dopants or grain boundaries also redirect energy carriers and affect k. (Fig. 2.7)

TABLE 2.1 Mean free path and critical film thickness for various materials at $T \approx 300$ K [3,4]

Material	λ_{mfp} (nm)	$L_{\mathrm{crit},y}$ (nm)	$L_{\mathrm{crit},x}$ (nm)
Aluminum oxide	5.08	36	22
Diamond (IIa)	315	2200	1400
Gallium arsenide	23	160	100
Gold	31	220	140
Silicon	43	290	180
Silicon dioxide	0.6	4	3
Yttria-stabilized zirconia	25	170	110



The Fluid State:

gas: $$k = \frac{1}{3} c_v \rho \bar{c} \lambda_{\text{mfp}} \qquad (2.10)$$

Since $\bar{c} \propto \sqrt{T}$ and $1/\sqrt{\mathcal{M}}$ (Fig. 2.8)

$$\rho \propto P \text{ and } \lambda_{\text{mfp}} \propto 1/P$$

$$\rightarrow \quad k \propto \sqrt{T} \quad \text{and} \quad k \text{ independent of } P$$

The Fluid State: Micro- and Nanoscale Effects

As in the solid state, k may be modified for small values of L/λ_{mfp} , when the fluid is contained in an small physical dimension.

In general:

- k depends strongly on temperature and is nearly independent of pressure.
- $k_{\text{solid}} > k_{\text{liquid}} > k_{\text{gas}}$

From Cengel, Heat Transfer, 2003.

TABLE 1–1

The thermal conductivities of some materials at room temperature

Material	k, W/m · °C
Diamond	2300
Silver	429
Copper	401
Gold	317
Aluminum	237
Iron	80.2
Mercury (l)	8.54
Glass	0.78
Brick	0.72
Water (l)	0.613
Human skin	0.37
Wood (oak)	0.17
Helium (g)	0.152
Soft rubber	0.13
Glass fiber	0.043
Air (g)	0.026
Urethane, rigid foam	0.026

2.2.2 Other Relevant Properties

Thermophysical properties:

- ***Transport properties***: k thermal conductivity (for heat transfer), ν, kinematic viscosity (for momentum transfer), D, mass diffusivity (for mass transfer)
  - thermal diffusivity, α --the ability of a material to conduct thermal energy relative to its ability to store thermal energy.

$$\alpha = \frac{k}{\rho c_p} \ [\mathrm{m}^2/\mathrm{s}, \text{ in the same units as } \nu \text{ and } D]$$

- ***Thermodynamic properties***: such as temperature T, pressure p, density ρ, and specific heat c_p, etc.

2.3 The Heat Diffusion Equation

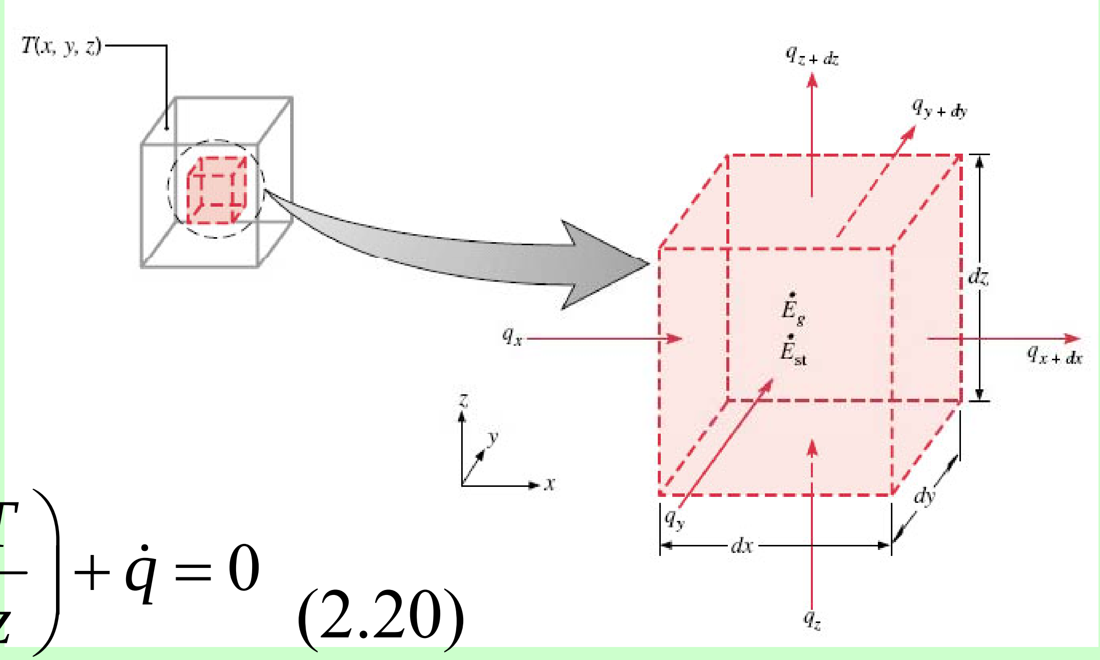
Performing energy conservation on a differential control volume (Figs. 2.11-2.13), the heat equation can be written as:

In Cartesian coordinates:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \tag{2.17}$$

If k =constant,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$



If steady-state,

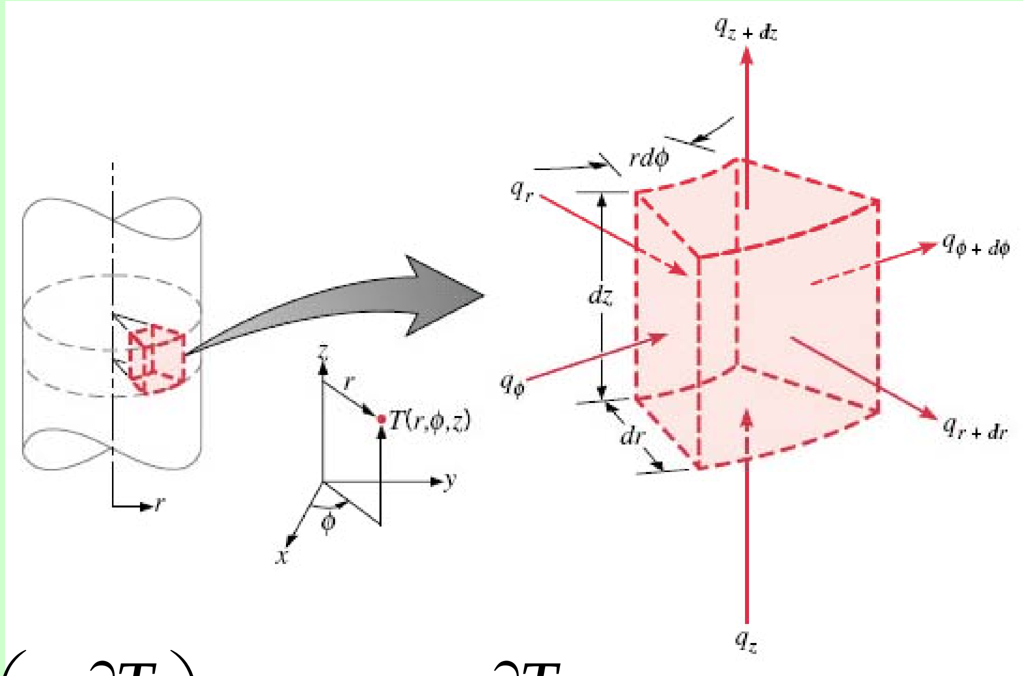
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0 \tag{2.20}$$

If one-dimensional with no energy generation,

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)=0 \qquad (2.21)$$

→under steady-state, one-dimensional conditions with no energy generation, the heat flux is a constant in the direction of transfer ($dq_x''/dx = 0$).

<u>**In cylindrical coordinates**</u>:



$$\vec{\boldsymbol{q}}'' = -k\nabla T$$

$$= -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial \phi} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

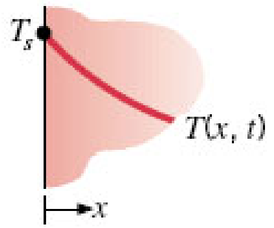
$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.24)$$

2.4 Boundary and Initial Conditions

TABLE 2.2 Boundary conditions for the heat diffusion equation at the surface ($x = 0$)

1. Constant surface temperature

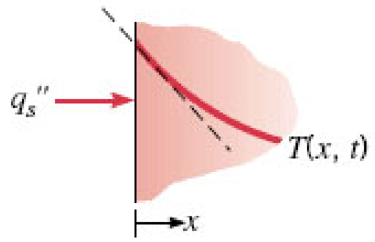
$$T(0, t) = T_s \tag{2.29}$$



2. Constant surface heat flux

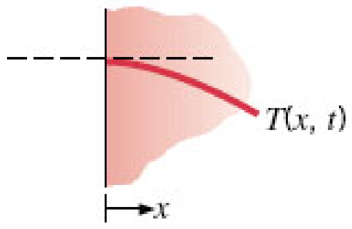
(a) Finite heat flux

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_s'' \tag{2.30}$$



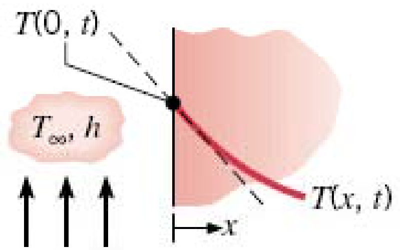
(b) Adiabatic or insulated surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0 \tag{2.31}$$



3. Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \tag{2.32}$$



EX 2.3