Chapter 16 Plane Motion of Rigid Bodies, Forces and Acceleration

16.1 Introduction

kinetics of rigid bodies : relation between the forces acting on the body, the shape and mass of body and the motion produced

in chapter, the relations between force and motion of system of particles are obtained

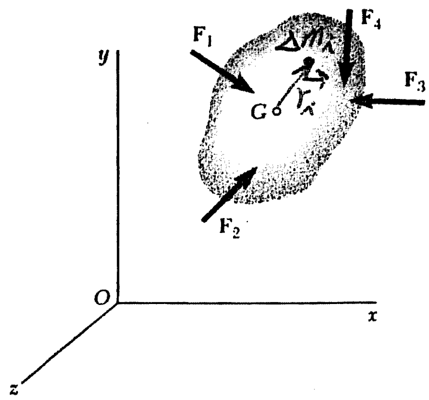
$$\Sigma \boldsymbol{F} = m\,\underline{\boldsymbol{a}}$$
$$\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$

in this chapter, the discussion will be limited to

1. plane motion of rigid body
2. the rigid body considered will consist only of plane slab, and symmetrical with respect to reference plane

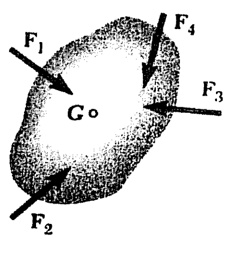
16.2 Equation of Motion for a Rigid Body

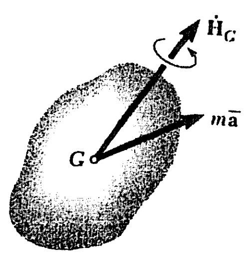
assume the rigid body is made of a large number n of particles of mass Δm_i, and subjected to several external forces, then



$$\Sigma \boldsymbol{F} = m\,\underline{\boldsymbol{a}}$$
and $$\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$

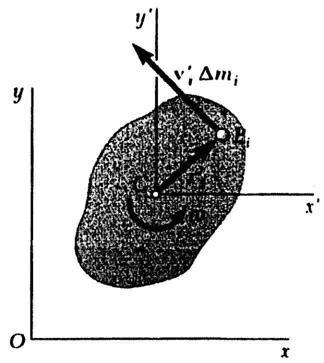
the system of external forces is equipollent to the system consisting of the vector $m\underline{a}$ attached at G and the couple of moment $\dot{\boldsymbol{H}}_G$





16.3 Angular Momentum of a Rigid Body in Plane Motion

consider a rigid slab in plane motion



$$\boldsymbol{H}_{G'} = \sum_{i=1}^{n} (\boldsymbol{r}_{i'} \times \Delta m_i \, \boldsymbol{v}_{i'})$$

$$\boldsymbol{v}_{i'} = \omega \times \boldsymbol{r}_{i'}$$

$$\therefore \quad \boldsymbol{H}_{G'} = \sum_{i=1}^{n} [\boldsymbol{r}_{i'} \times (\omega \times \boldsymbol{r}_{i'}) \Delta m_i]$$

$\boldsymbol{H}_{G'}$ has the same direction with ω

$$\because \quad \omega \perp \boldsymbol{r}_{i'} \qquad \therefore \; |\omega \times \boldsymbol{r}_{i'}| = \omega\, r_{i'}$$

and $\boldsymbol{r}_{i'} \perp (\omega \times \boldsymbol{r}_{i'}) \qquad \therefore \; |\boldsymbol{r}_{i'} \times (\omega \times \boldsymbol{r}_{i'})| = \omega\, r_{i'}^{2}$

thus $|\boldsymbol{H}_{G'}| = \omega \sum_{i=1}^{n} r_{i'}^{2} \Delta m_i = \omega \int r'^2 \, dm$

$$H_{G'} = \underline{I}_{mass}\, \omega$$

and $\underline{I}_{mass} = \rho\, t\, \underline{I}_{area}$ for plane slab

where $\underline{I}$ is the moment of inertia of the slab about its centrodial axis

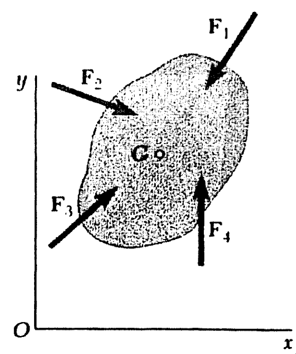
then $\dot{\boldsymbol{H}}_{G'} = \underline{I}\, \dot{\omega} = \underline{I}\, a = \dot{\boldsymbol{H}}_{G}$ [from chapter 14]

note that $\underline{I}$ doesn' t change during the plane motion of the rigid body thus it is obtained

$$\Sigma \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G = \underline{I}\, a$$

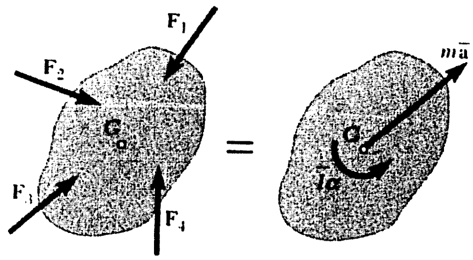
16.4 Plane Motion of a Rigid Body, D'Alembert's Principle

consider a rigid body of mass m acted upon by several external forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots$, the relation between forces and motions are



$$\Sigma F_x = m\, a_x \qquad \Sigma F_y = m\, a_y$$

and $\Sigma M_G = \dot{H}_G = \underline{I}\,\alpha$



thus the motion of the slab is completely defined by the resultant and moment resultant about G of the external forces acting on it

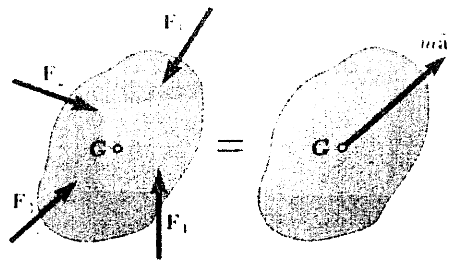
two system of forces has same resultant and same moment resultant are called equipollent, they are also equivalent when acting on a rigid body, because they have exactly the same effect on a given rigid body

the external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body

translation :

$\Sigma \boldsymbol{F} \neq 0 \qquad \underline{a} \neq 0$

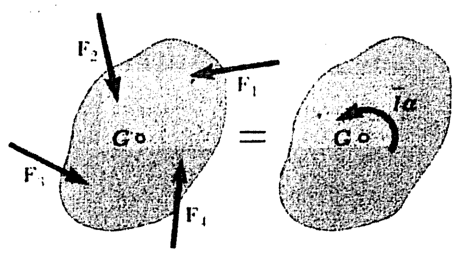
$\Sigma \boldsymbol{M}_G = 0 \qquad \alpha = 0$



centroidal rotation :

$\Sigma \boldsymbol{F} = 0 \qquad \underline{a} = 0$

$\Sigma \boldsymbol{M}_G \neq 0 \qquad \alpha \neq 0$



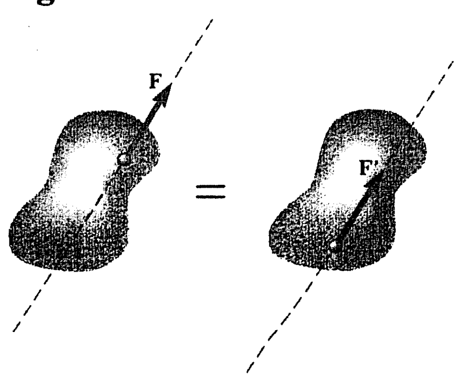
general plane motion :

$\Sigma \boldsymbol{F} \neq 0 \qquad \underline{a} \neq 0$

$\Sigma \boldsymbol{M}_G \neq 0 \qquad \alpha \neq 0$

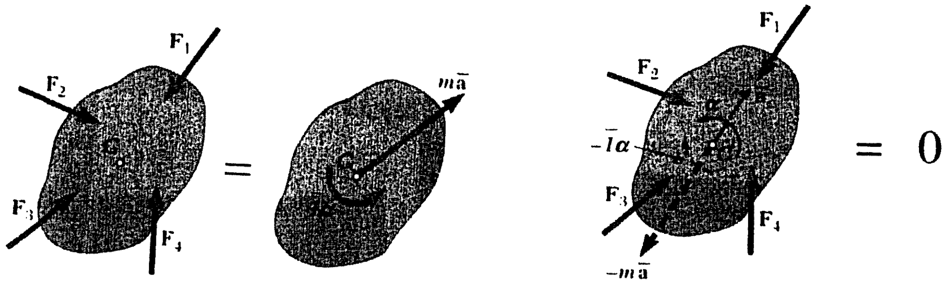
16.5 A Remark on the Axioms of the Mechanics of Rigid Bodies

from the principle of transmissibility, two forces F and F' have the same effect on the rigid body



16.6 Solution of Problems Involving the Motion of a Rigid Body

there are three algebraic equations can be solved problem of plane motion, free body diagrams of the dynamic equilibrium can be applied



the use of free body diagrams have the following advantages :

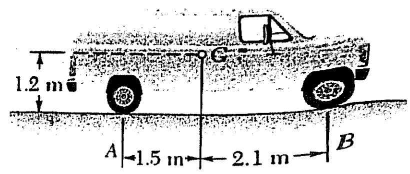
1. much clear understanding of the effect of forces on the motion of the body
2. this approach makes the dynamic problem into vector problem
3. unified approach for the analysis of plane motion of rigid body
4. resolve the plane motion of a rigid body into a translation and a centroidal rotation
5. may be extended to the study of the general three-dimensional motion

16.7 System of Rigid Bodies

a diagram similar to pervious section may be drawn for each part of the system, the equation of motion obtained from these diagrams are solved simultaneously

Sample Problem 16.1

$v_0 = 10$ m/s weight $= W$



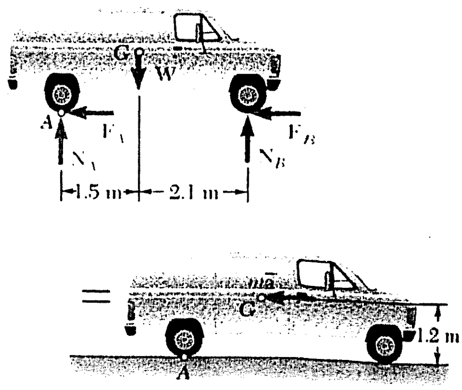
uniformly accelerated motion when the break applied, and the wheels stop rotating

the truck skidded to rest in 7 m

determine the reactions and friction at each wheel for uniformly accelerated motion

$$0 = \underline{v}^2 = \underline{v}_0^2 + 2\,\underline{a}\,x = 10^2 + 2\,\underline{a} \times 7$$

$$\underline{a} = -\,7.14 \text{ m/s}^2 \leftarrow$$



$$\Sigma F_y = (\Sigma F_y)_{eff} = 0$$

$$N_A + N_B - m\,g = 0$$

$$\Sigma F_x = (\Sigma F_x)_{eff}$$

$$-F_A - F_B = -m\,\underline{a}$$

$$-\mu\,(N_A + N_B) = -m\,\underline{a}$$

$$-\mu\,m\,g = -m\,\underline{a}$$

$$\mu = \underline{a}\,/\,g = 7.14\,/\,9.81 = 0.728$$

$$\Sigma M_A = (\Sigma M_A)_{eff}$$

$$-W \times 1.5 + N_B \times 3.6 = W/g \times 7.14 \times 1.2$$

$$N_B = 0.659\,W$$

$N_A + N_B = W$ $\qquad$ $N_A = 0.341\,W$

$F_A = \mu_k\,N_A = 0.248\,W$ $\qquad$ $F_B = \mu_k\,N_B = 0.48\,W$

$N_{front} = ½\,N_B = 0.3295\,W$ $\qquad$ $F_{front} = ½\,F_B = 0.24\,W$

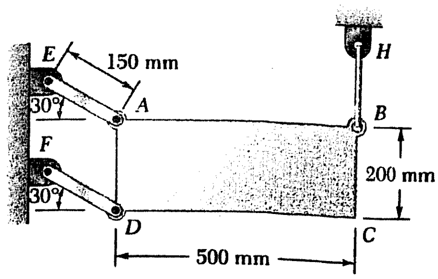
$N_{rear} = ½\,N_A = 0.1705\,W$ $\qquad$ $F_{rear} = ½\,F_A = 0.124\,W$

Sample Problem 16.2

$m = 8$ kg

for AB sudden break, determine $\underline{a}$, F_{AE} and F_{DF}

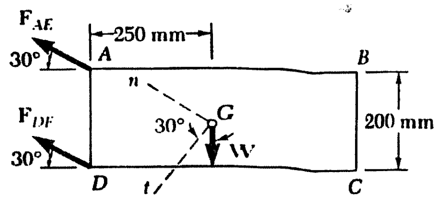
for the cable AB is cut, the plate moves in curvilinear translation



$$\Sigma F_t = (\Sigma F_t)_{eff}$$

$$m\,g \cos 30° = m\,\underline{a} \qquad (1)$$

$a = g \cos 30^o = 8.5 \text{ m/s}^2$ ↙| 30°



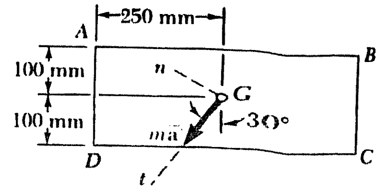
$\Sigma F_n = (\Sigma F_n)_{eff}$

$F_{AE} + F_{DF} - m g \sin 30^o = 0$ (2)

$\Sigma M_G = (\Sigma M_G)_{eff}$

$(F_{AE} + F_{DF}) \sin 30^o (250) + (F_{DF} - F_{AE}) \cos 30^o (100) = 0$

$F_{DF} = -0.1815 F_{AE}$ (3)

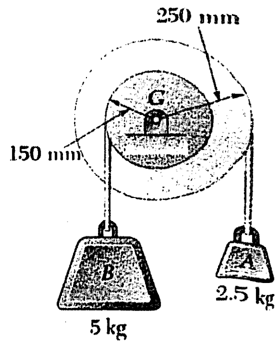


substituting (3) into (2)

$F_{AE} = 0.6109 m g = 47.9$ N (T)

$F_{DF} = -0.1109 m g = 8.7$ N (C)

Sample Problem 16.3



$m_{pulley} = 6$ kg no axle friction

$k = 200$ m (radius of gyration)

determine a, a_A and a_B

for equilibrium of the system, if $m_A = 2.5$ kg

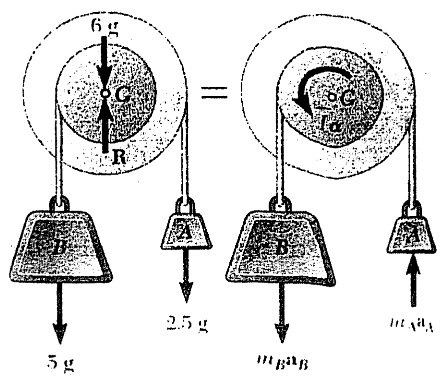
$\Sigma M_G = 0 \quad W_B \times 150 = W_A \times 250$

$m_B g \times 150 = 2.5 g \times 250$

$m_B = 4.167$ kg

but $m_B = 5 \text{ kg} > 4.167$ kg

thus the system will rotate counterclockwise



$a_A = 0.5\, a$ ↑ $\quad a_B = 0.3\, a$ ↓

$I_p = m k^2 = 6 \times 0.2^2 = 0.24$ kg-m²

$\Sigma M_G = (\Sigma M_G)_{eff}$

$W_B \times 0.15 - W_A \times 0.25 = I_p\, a + m_B\, a_B \times 0.15 + m_A\, a_A \times 0.25$

$$5g \times 0.15 - 2.5g \times 0.25 = 0.24a + 5 \times 0.15a \times 0.15 + 2.5 \times 0.25a \times 0.25$$

$a = 2.41 \text{ rad/s}^2$ (CCW)

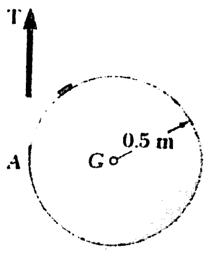
$a_A = r_A\, a = 0.6.3 \text{ m/s}^2 \uparrow$

$a_B = r_B\, a = 0.362 \text{ m/s}^2 \downarrow$

Sample Problem 16.4

consider a circular disk $m = 15$ kg, $r = 0.5$ m

$T = 180$ N, determine a_G, a and a_{cord}

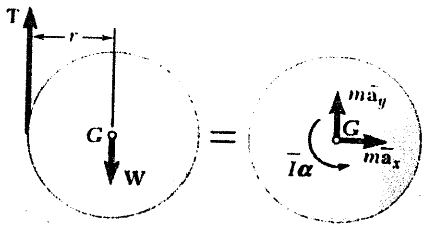


the disk is in general plane motion

$\Sigma F_x = (\Sigma F_x)_{eff}$

$0 = m\, a_x \qquad a_x = 0$

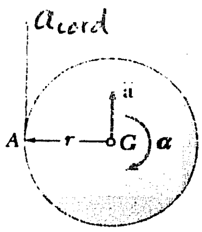
$\Sigma F_y = (\Sigma F_y)_{eff}$



$T - W = m\, a_y$

$a_y = 2.19 \text{ m/s}^2$

$\Sigma M_G = (\Sigma M_G)_{eff}$



$T r = I a = \frac{1}{2} m r^2 a$

$a = -2\, T / m\, r = -48 \text{ rad/s}^2$

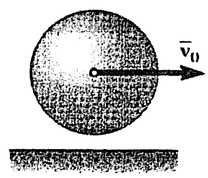
$a_{cord} = (a_A)_t = a_G + (a_{A/G})_t$

$= 2.19 + 0.5 \times 48 = 26.2 \text{ m/s}^2$

Sample Problem 16.5

mass of sphere : m friction coefficient : μ_k

initial velocity : v_0



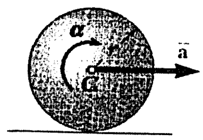
determine (a) t_1 at which the sphere will start rolling without sliding

(b) v and ω of the sphere at t_1

when the sphere with both rolling and sliding

$\Sigma F_y = (\Sigma F_y)_{eff}$ $\quad N - W = 0$

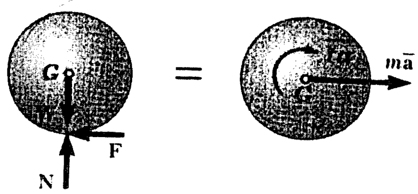
$N = W = m g$



and $\quad F = \mu_k N = \mu_k m g$

$\Sigma F_x = (\Sigma F_x)_{eff}$ $\quad -F = m a \quad a = -\mu_k g \leftarrow$

$\Sigma M_G = (\Sigma M_G)_{eff}$



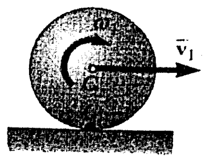
$F r = I \alpha \quad \mu_k m g r = (2/5) m r^2 \alpha$

$$\alpha = \frac{2}{5} \frac{\mu_k g}{r}$$

$t = 0 \quad v = v_0 \qquad v = v_0 + a t = v_0 - \mu_k g t$

$$t = 0 \quad \omega_0 = 0 \qquad \omega = \omega_0 + \alpha t = \frac{5}{2} \frac{\mu_k g}{r} t$$

at time t_1, the sphere start rolling without sliding



$v_C = 0$ i.e. $v_1 = r \omega_1$

$$v_0 - \mu_k g t_1 = \frac{5}{2} \frac{\mu_k g}{r} t_1 r$$

$$t_1 = \frac{2}{7} \frac{v_0}{\mu_k g}$$

and $$\omega_1 = \frac{5}{2} \frac{\mu_k g}{r} t_1 = \frac{5}{7} \frac{v_0}{r} \quad \text{clockwise}$$

$$v_1 = r \omega_1 = 5 v_0 / 7 \rightarrow$$

when the sphere start rolling, no relative velocity between sphere and surface, $F \neq \mu_k m g$, $F \simeq 0$ (due to rolling resistance effect)

then $\quad a = 0 \qquad \alpha = 0$

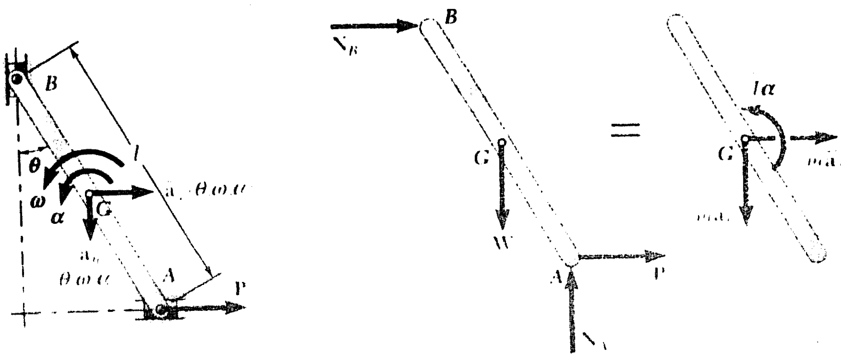
$\omega = \omega_1 = $ constant

$v = v_1 = $ constant

16.8 Constrained Plane Motion

consider the rigid body moves under given constrains, the relations between $\boldsymbol{v}$, ω, $\underline{\boldsymbol{a}}$ and α can be found by kinematics analysis

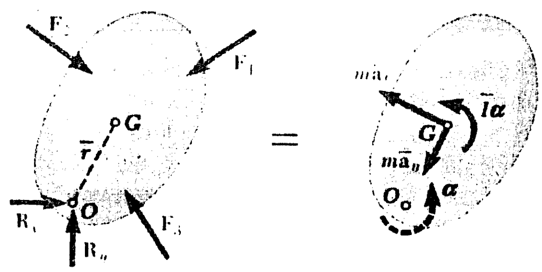
consider a slender rod AB of length l and mass m, the rod is pulled by a force $\boldsymbol{P}$ at A, then the acceleration $\underline{\boldsymbol{a}}$ and α can be determined at any given instant



when a mechanism consists of several moving parts, the approach just described can be used with each part of the mechanism

two particular cases are considered

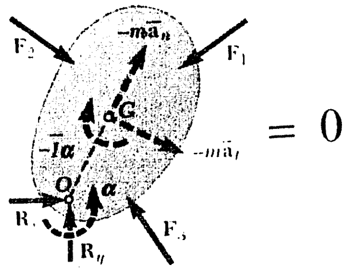
1. noncentroidial rotation



$$\underline{a}_t = \underline{r}\,\alpha$$

$$\underline{a}_n = \underline{r}\,\omega^2$$

since line OG belongs to the body, its ω and α also the same as the body



$$\Sigma M_0 = \underline{I}\,\alpha + (m\,\underline{r}\alpha)\,\underline{r} = (\underline{I} + m\underline{r}^2)\,\alpha$$

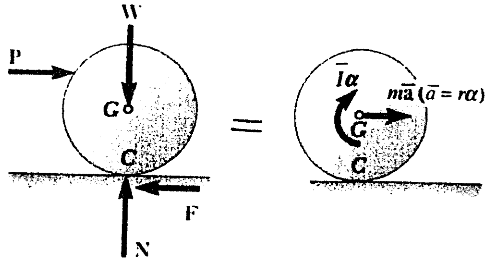
but $\underline{I}_0 = \underline{I} + m\underline{r}^2$ [parallel-axis theorem]

thus $\Sigma M_0 = \underline{I}_0\,\alpha$

it does not mean that the system of external forces is equivalent to a couple of moment $\underline{I}_0\,\alpha$, other effective forces occur

for ω = constant, $\alpha = 0$, $\underline{a}_t = 0$, only $\underline{a}_n$ exists

2. rolling motion



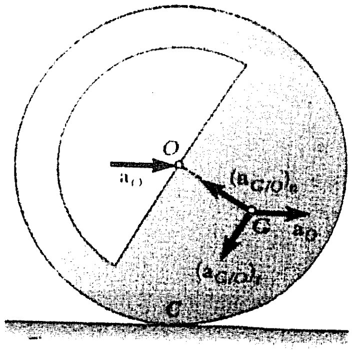
for a plane disk is rolling without sliding, then $\underline{a}$ and α are not independent

$$\underline{a} = r\,\alpha$$

when a disk is in rolling motion, three cases can be summarized:

rolling, no sliding :	$F \leqq \mu_s N$	$\underline{a} = r\,\alpha$
rolling, sliding impending :	$F = \mu_s N$	$\underline{a} = r\,\alpha$
rolling and sliding :	$F = \mu_s N$	$\underline{a}$ and α independent

for an unbalanced disk, O and G are not coincided, when the disk in rolling without sliding



$$\underline{a} \neq r\,\alpha$$

but $\quad a_0 = r\,\alpha$

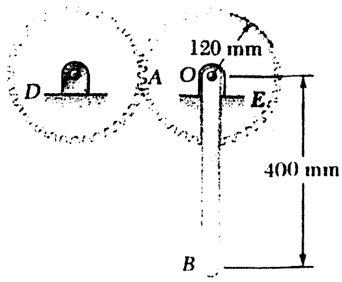
and $\quad \underline{\boldsymbol{a}} = \boldsymbol{a}_0 + \boldsymbol{a}_{G/O}$

$$= \boldsymbol{a}_0 + (\boldsymbol{a}_{G/O})_t + (\boldsymbol{a}_{G/O})_n$$

where $\quad (a_{G/O})_t = (OG)\,\alpha$

$$(a_{G/O})_n = (OG)\,\omega^2$$

Sample Problem 16.6



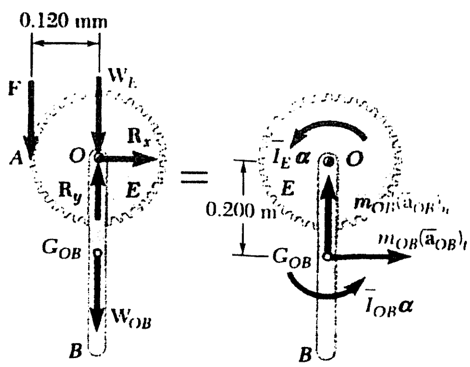
$\omega = 8$ rad/s clockwise

$\alpha = 40$ rad/s^2 counterclockwise

$m_{OB} = 3$ kg $\qquad m_E = 4$ kg $\quad k_E = 85$ mm

determine (a) the tangential force by D on E

(b) the reactions at O



$(\underline{a}_{OB})_t = \underline{r}\,\alpha = 0.2 \times 40 = 8$ m/s$^2 \rightarrow$

$(\underline{a}_{OB})_n = \underline{r}\,\omega^2 = 0.2 \times 8^2 = 12.8$ m/s^2 $\uparrow$

$\underline{I}_E\,\alpha = 4\,(0.085)^2 \times 40 = 1.156$ N-m

$\underline{I}_{OB}\,\alpha = (1/12)\,3 \times 0.4^2 \times 40 = 1.6$ N-m

$\Sigma M_0 = (\Sigma M_0)_{eff}$

$F \times 0.12 = \underline{I}_E \, a + m_{OB} (\underline{a}_{OB})_t \times 0.2 + \underline{I}_{OB} \, a$

$= 1.156 + 3 \times 8 \times 0.2 + 1.6$

$F = 63 \text{ kN} \downarrow$

$\Sigma F_x = (\Sigma F_x)_{eff}$

$R_x = m_{OB} (\underline{a}_{OB})_t = 3 \times 8 = 24 \text{ N} \rightarrow$

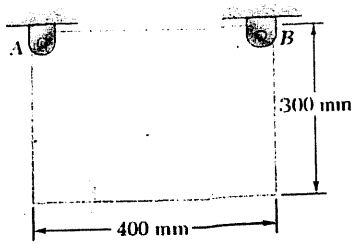
$\Sigma F_y = (\Sigma F_y)_{eff}$

$R_y - F - W_E - W_{OB} = m_{OB} (\underline{a}_{OB})_n$

$R_y - 63 - 4 \times 9.81 - 3 \times 9.81 = 3 \times 12.8$

$R_y = 170 \text{ N} \uparrow$

Sample Problem 16.7



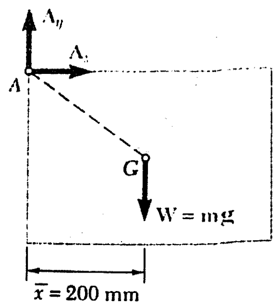
$m = 30$ kg if pin B is suddenly removed

determine a and R_A

the plate is rotating about A

$\because \quad \omega = 0 \quad => \quad a_n = 0$

then $\quad \underline{a} = \underline{r} \, a$



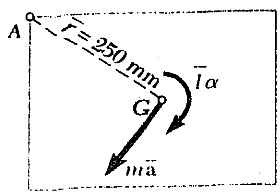
$\Sigma M_A = (\Sigma M_A)_{eff}$

$m \, g \, \underline{x} = \underline{I} \, a + m \, \underline{a} \, \underline{r} = (\underline{I} + m \, \underline{r}^2) \, a$

then $\quad a = \dfrac{m \, g \, \underline{x}}{\underline{I} + m \, \underline{r}^2}$

$\underline{I} = (m/12)(a^2 + b^2) = 0.625 \text{ kg-m}^2$

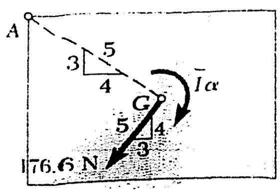
=



for $\quad m = 30$ kg $\quad \underline{x} = 0.25$ m $\quad \underline{r} = 0.25$ m

$a = 23.54 \text{ rad/s}^2$ clockwise

=



and $\quad m \, \underline{a} = m \, \underline{r} \, a = 176.6$ N

$\Sigma F_x = (\Sigma F_x)_{eff}$

$A_x = -(3/5)(176.6) = -106\text{ N} \leftarrow$

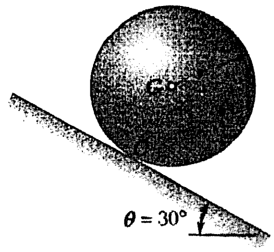
$\Sigma F_y = (\Sigma F_y)_{eff}$

$A_y - mg = -(4/5)(176.6)$

$A_y = 153\text{ N} \uparrow$

Sample Problem 16.8

$\theta = 30^o$ sphere has no initial velocity,



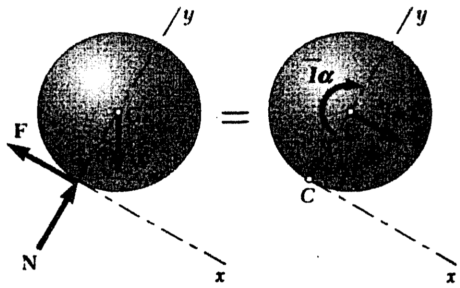
rolls without sliding, determine

(a) minimum μ_s for rolling without sliding

(b) $\underline{v}$ after rolled 3 m

(c) $\underline{v}$ after rolled 3 m if no friction

(a) $\because$ no sliding, $\therefore\ a = r\alpha$



$\Sigma M_C = (\Sigma M_C)_{eff}$

$(mg \sin\theta)\, r = (ma)\, r + I\alpha$

$= m\, r\alpha\, r + (2/5)\, mr^2\, \alpha$

$$\alpha = \frac{5\, g \sin\theta}{7\, r}$$

$$a = r\alpha = \frac{5\, g \sin\theta}{7} = \frac{5 \text{ x } 9.81 \text{ x } 0.5}{7} = 3.5\text{ m/s}^2$$

$\Sigma F_x = (\Sigma F_x)_{eff}$ $\quad mg \sin\theta - F = m\, a = m\dfrac{5\, g \sin\theta}{7}$

$F = (2/7)\, mg \sin\theta = 0.143\, mg$ ↖ 30^o

$\Sigma F_y = (\Sigma F_y)_{eff}$ $\quad N - mg \cos\theta = 0$

$N - mg \cos\theta = 0.866\, mg$ ↗ 60^o

$$\mu_s = \frac{F}{N} = \frac{0.143\, mg}{0.866\, mg} = 0.165$$

(b) $\quad a = 3.5\text{ m/s}^2 = \text{constant}$

$v^2 = v_0^2 + 2\,a\,(x - x_0) = 0 + 2 \times 3.5 \times 3$

$v = 4.59$ m/s

(c) for $\mu_s = 0$ $\quad F = 0$

$\Sigma M_G = (\Sigma M_G)_{eff} = 0 \qquad a = 0$

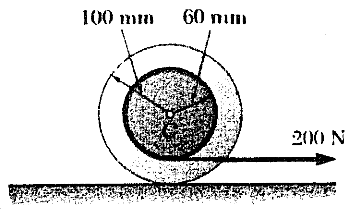
$\Sigma F_x = (\Sigma F_x)_{eff} \qquad mg \sin\theta = m\,a$

$a = 4.905$ m/s^2 $\quad$ [a and a independent]

$v^2 = v_0^2 + 2\,a\,(x - x_0) = 0 + 2 \times 4.905 \times 3$

$v = 5.42$ m/s

Sample Problem 16.9

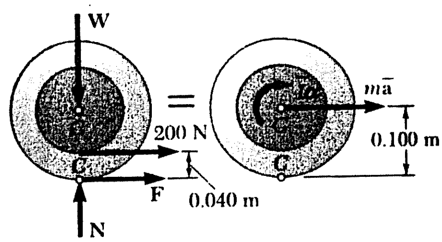


$\mu_s = 0.2 \qquad \mu_k = 0.15$

$m = 50$ kg $\qquad k = 70$ mm

determine a and a of the drum

(a) assume the drum rolling without sliding



$a = r\,a = 0.1\,a$

$I = m\,k^2 = 50 \times 0.07^2 = 0.245$ kg-m^2

$\Sigma M_C = (\Sigma M_C)_{eff}$

$200 \times 0.04 = m\,a \times 0.1 + I\,a$

$a = 10.74$ rad/s^2 clockwise

$a = 1.074$ m/s^2 $\rightarrow$

$\Sigma F_x = (\Sigma F_x)_{eff}$

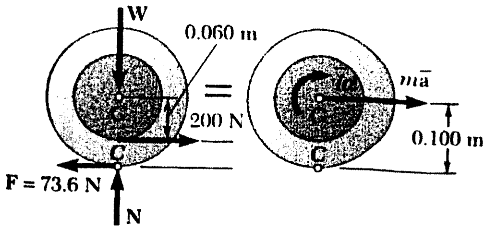
$200 + F = m\,a \qquad F = -146.3$ N $\leftarrow$

$\Sigma F_y = (\Sigma F_y)_{eff}$

$N = W = m\,g = 490.5$ N

thus $F_{max} = \mu_s N = 0.2 \times 490.5 = 98.1$ N then $|F| > F_{max}$, it is impossible for rolling without sliding

(b) rolling and sliding



$$F = \mu_k N = 0.15 \times 490.5 = 73.6 \text{ N}$$

$$\Sigma F_x = (\Sigma F_x)_{eff}$$

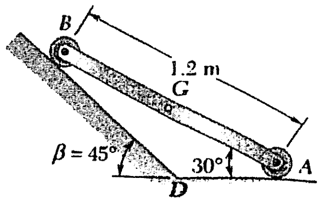
$$200 - 73.6 = m\,a \qquad a = 2.53 \text{ m/s}^2$$

$$\Sigma M_G = (\Sigma M_G)_{eff}$$

$$73.6 \times 0.1 - 200 \times 0.06 = 0.245\, a$$

$$a = -18.49 \text{ rad/s}^2 \text{ counterclockwise}$$

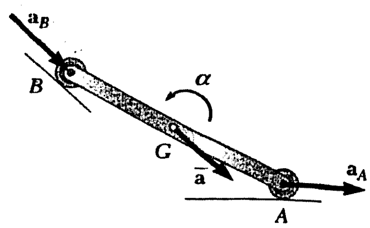
Sample Problem 16.10



$m = 25$ kg $\qquad l = 1.2$ m

the rod is released form rest, no friction

determine a and reactions at A and B



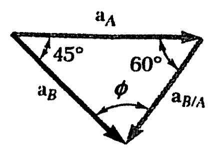
$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$$

$\because\ \omega = 0$, $\boldsymbol{a}_{B/A}$ has no normal component

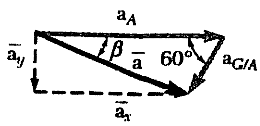
$$\boldsymbol{a}_{B/A} = (\boldsymbol{a}_{B/A})_t = 1.2\,a \angle 60^\circ$$

$$\boldsymbol{a}_A = a_A \rightarrow$$

$$\boldsymbol{a}_B = a_B \searrow 45^\circ$$



from the vector polygon, it can be obtained



$$a_A = 1.64\,a \qquad a_B = 1.47\,a$$

$(\boldsymbol{a}_{B/A})_t = 1.2\,a \angle 60^\circ$ and a is counterclockwise

the acceleration of center is now obtained

$$\boldsymbol{a} = \boldsymbol{a}_G = \boldsymbol{a}_A + \boldsymbol{a}_{G/A}$$

$$= 1.64\,a\,[\rightarrow] + 0.6\,a\,[\swarrow 60^\circ]$$

then $a_x = a_A - a_{G/A} \cos 60^o = 1.34\, a \rightarrow$

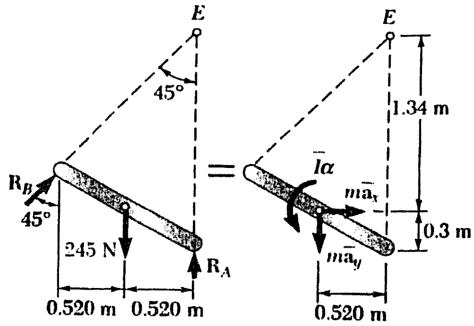
$a_y = a_{G/A} \sin 60^o = 0.52\, a \downarrow$

$I = (1/12)\, m\, l^2 = (1/12)\, 25 \times 1.2^2 = 3$ kg-m^2

$I\, a = 3\, a$

$m\, a_x = 25 \times 1.34\, a = 33.5\, a$

$m\, a_y = 25 \times 0.52\, a = 13\, a$



equations of motion

$\Sigma M_E = (\Sigma M_E)_{eff}$

$25\, g \times 0.52 = m\, a_x \times 1.34 + m\, a_y \times 0.52 + I\, a$

$a = 2.33$ rad/s^2 counterclockwise

$\Sigma F_x = (\Sigma F_x)_{eff}$

$R_B \sin 45^o = m\, a_x = 33.5 \times 2.33 = 78.1$

$R_B = 110$ N $\nearrow 45^o$

$\Sigma F_y = (\Sigma F_y)_{eff}$

$R_A + R_B \cos 45^o - 25\, g = -\, m\, a_y = -\ 13 \times 2.33$

$R_A = 136.6$ N $\uparrow$