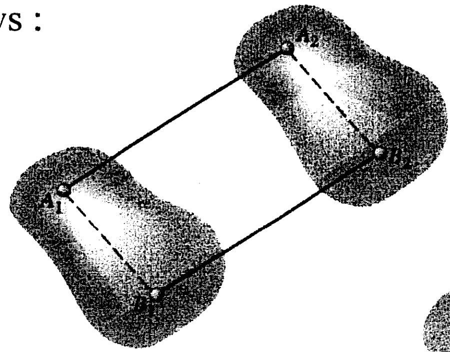
Chapter 15 Kinematics of Rigid Bodies

15.1 Introduction

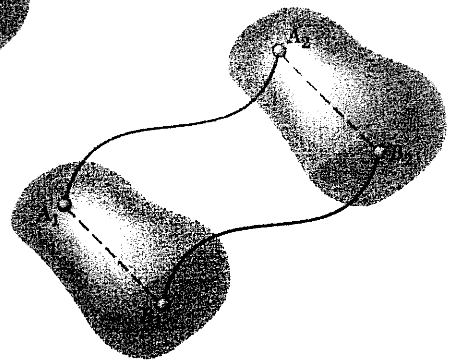
relationship between time, positions, velocities and accelerations of rigid bodies motion may be grouped as follows :

1. translation : if any straight line inside the body deeps the same direction during the motion



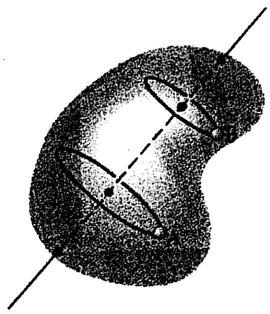
rectilinear translation : if the paths of motion of the particles are straight line

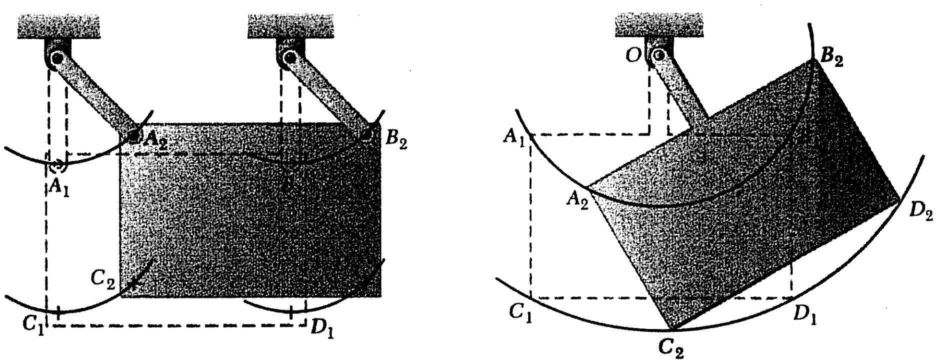
curvilinear translation : if the paths of motion of the particles are curved line



2. rotation about a fixed axis : AA' is called the axis of rotation, $\boldsymbol{v}$ and $\boldsymbol{a}$ are zero if the particle located on this axis

it is different from curvilinear translation



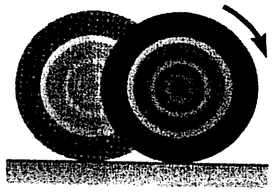


(*a*) Curvilinear translation (*b*) Rotation

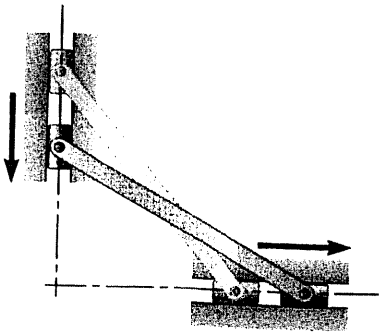
3. general plane motion :

plane motion : each particle moves in a given plane

any plane motion which is neither a rotation nor translation is referred to as a general motion

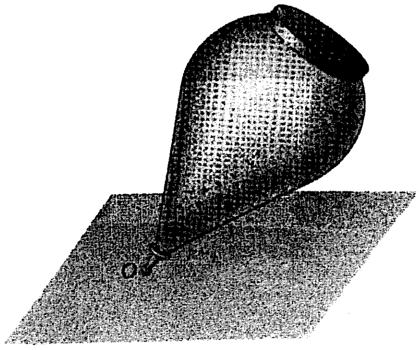


(a) Rolling wheel



(b) Sliding rod

4. motion about a fixed point : the three-dimensional motion of a rigid body attached at a fixed point



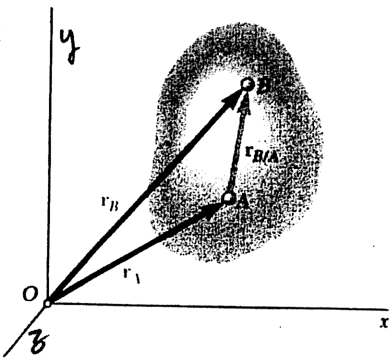
5. general motion : any rigid motion does not fall in any of above

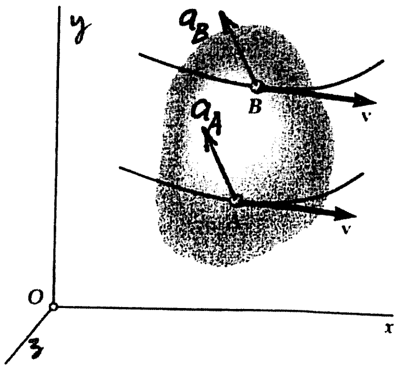
15.2 Translation

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

$\because$ $\boldsymbol{r}_{B/A}$ must remain constant direction and magnitude

$\therefore$ $\boldsymbol{r}_{B/A}$ = constant





differentiate with respect to time, then

$$\dot{\boldsymbol{r}}_B = \dot{\boldsymbol{r}}_A \quad \text{or} \quad \boldsymbol{v}_B = \boldsymbol{v}_A$$

also $$\boldsymbol{a}_B = \boldsymbol{a}_A$$

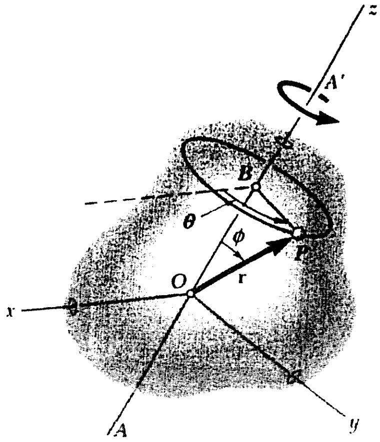
that means when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant

15.3 Rotation About a Fixed Axis

consider a rigid body rotates about a fixed axis AA', and let P be a point of the body and $\boldsymbol{r}$ its position vector

let B be the projection of P on AA', then

$$BP = r \sin \phi = \text{constant}$$



P will describe a circle of center at B

denote the angle θ the angular coordinate of the body, it is defined the angle between $\boldsymbol{BP}$ and xz plane

$$1 \text{ revolution} = 2\pi \text{ rad} = 360^{\circ}$$

$$\boldsymbol{v} = \frac{\boldsymbol{dr}}{dt} \qquad v = \frac{ds}{dt}$$

$$\because \quad \Delta s = BP\,(\Delta\theta) = r \sin \phi\,(\Delta\theta)$$

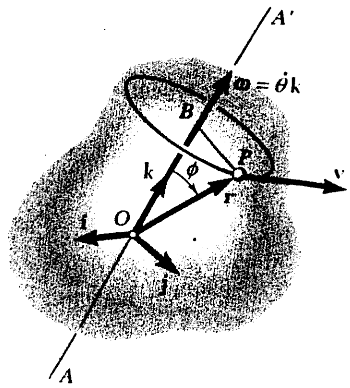
$$v = ds/dt = r \sin \phi\, \dot{\theta} \qquad \dot{\theta} = d\theta/dt$$

θ depends on the position of P, but $\dot{\theta}$ is independent of P

$\boldsymbol{v}$ is a vector $\perp$ the plane containing $\boldsymbol{AA'}$ and $\boldsymbol{r}$

define $\quad \boldsymbol{\omega} = \dot{\theta}\,\boldsymbol{k}$

the angular velocity of the rigid body



$$\boldsymbol{v} = \boldsymbol{dr}/dt \equiv \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{a} = \boldsymbol{dv}/dt = d(\boldsymbol{\omega} \times \boldsymbol{r})/dt$$

$$= (d\boldsymbol{\omega}/dt) \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{dr}/dt)$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v}$$

where $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is called angular acceleration

$$\boldsymbol{\alpha} = \alpha\,\boldsymbol{k} = \dot{\omega}\,\boldsymbol{k} = \ddot{\theta}\,\boldsymbol{k}$$

$\boldsymbol{a}$ is the sum of two vectors

$\alpha \times \boldsymbol{r}$ is the tangent to the circle which is the tangential component of $\boldsymbol{a}$

$\omega \times (\omega \times \boldsymbol{r})$ is a vector triple product, $(\omega \times \boldsymbol{r})$ is tangent to the circle, and $\omega \times (\omega \times \boldsymbol{r})$ is directed to the center B, which is normal component of $\boldsymbol{a}$

consider a plane containing the circle of point P, choose xy plane as the reference plane

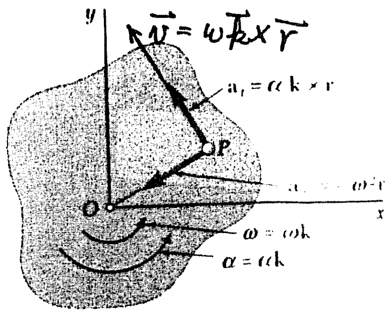
$$\boldsymbol{v} = \omega \times \boldsymbol{r}$$

and $v = r\,\omega$

$$\boldsymbol{a} = \alpha\,\boldsymbol{k} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_t + \boldsymbol{a}_n$$

where $\boldsymbol{a}_t = \alpha\,\boldsymbol{k} \times \boldsymbol{r}$ $\qquad a_t = \alpha\, r$

$\boldsymbol{a}_n = -\omega^2 \boldsymbol{r}$ $\qquad a_n = r\,\omega^2$



15.4 Equations Defining the Rotation of a Rigid Body About a Fixed Axis

the motion of rigid body rotating about a fixed axis is said to be known when $\theta(t)$ is known

$$\omega = d\theta / dt$$

$$\alpha = d\omega / dt = d^2\theta / dt^2$$

$$\alpha = d\omega / dt = (d\omega / d\theta)\,(d\theta / dt) = \omega\,(d\omega / d\theta)$$

two particular cases of rotation are encountered

1. uniform rotation

$\alpha = 0 \qquad \omega = \text{constant}$

then $\theta = \theta_0 + \omega\, t$

2. uniformly accelerated rotation

$\alpha = \text{constant}$

$\omega = \omega_0 + \alpha\, t \qquad \omega^2 = \omega_0^2 + 2\,\alpha\,(\theta - \theta_0)$

$\theta = \theta_0 + \omega_0\, t + \frac{1}{2}\,\alpha\, t^2$

Sample Problem 15.1

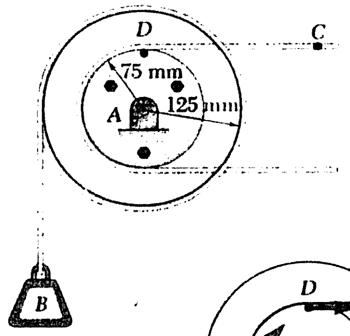
$v_C = 300$ mm/s $\quad a_C = 225$ mm/s^2

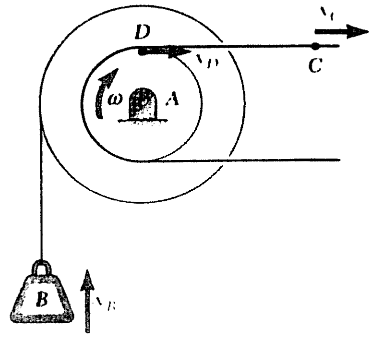
determine

(a) numbers of revolutions at $t = 2$ s

(b) v_B and Δy_B at $t = 2$ s

(c) $\boldsymbol{a}_D$ at $t = 0$





$(v_D)_0 = (v_C)_0 = 300$ mm/s $\rightarrow$

$(a_D)_t = a_C = 225$ mm/s^2 $\rightarrow$

$(v_D)_0 = r\,\omega_0 \qquad \omega_0 = 4$ rad/s

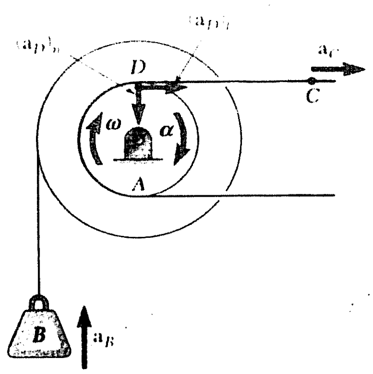
$(a_D)_t = r\,\alpha \qquad \alpha = 3$ rad/s^2

at $t = 2$ s

$\omega = \omega_0 + \alpha\,t = 4 + 3 \times 2 = 10$ rad/s

$\theta = \omega_0\,t + \frac{1}{2}\,\alpha\,t^2 = 4 \times 2 + \frac{1}{2}\,3 \times 2^2 = 14$ rad

no. of rev. $= 14 / 2\pi = 2.23$ rev.



$v_B = r\,\omega = 125 \times 10 = 1250$ mm/s $\uparrow$

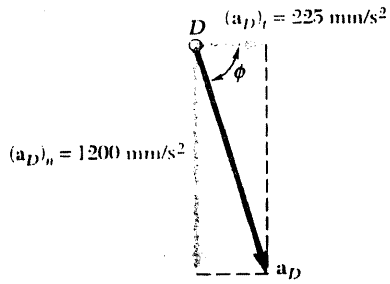
$\Delta y_B = r\,\theta = 125 \times 14 = 1750$ mm

at $t = 0$

$(a_D)_t = 225$ mm/s^2

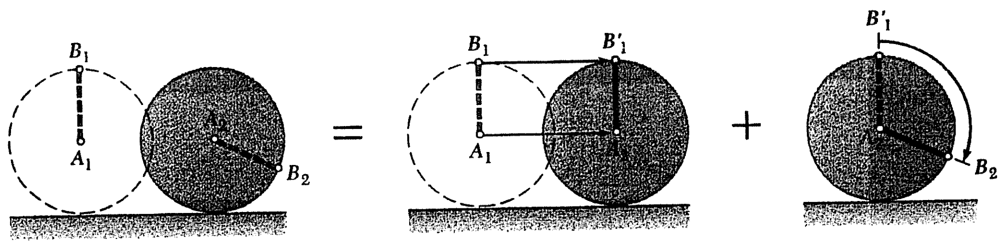
$(a_D)_n = r_D\,\omega^2 = 1200$ mm/s^2

i.e. $\quad \boldsymbol{a}_D = 1220$ mm/s^2 $\searrow$ 79.4°

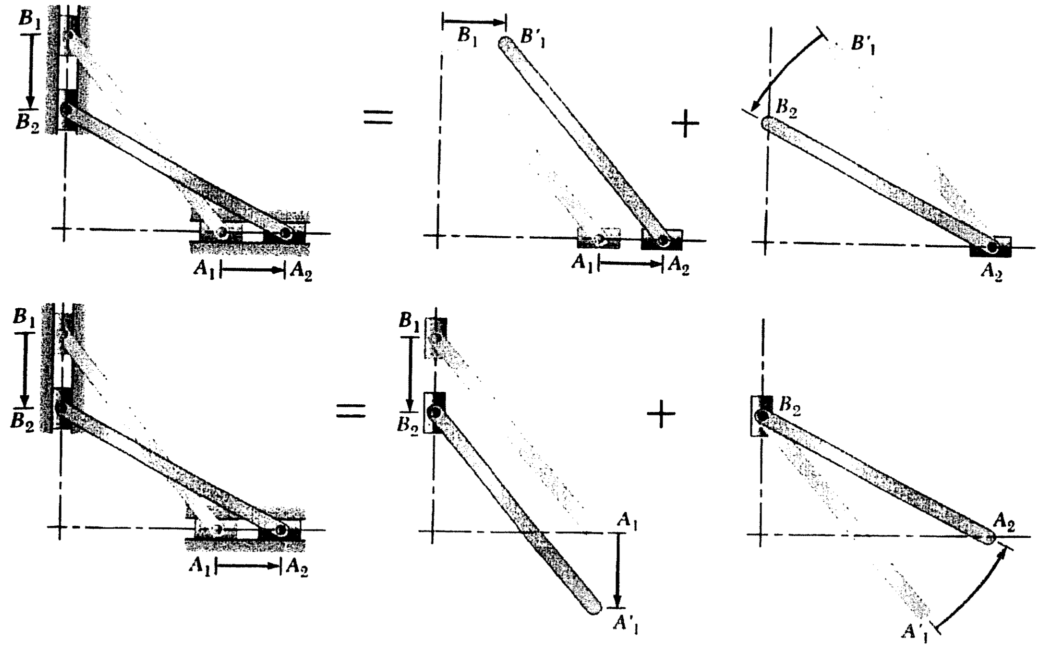


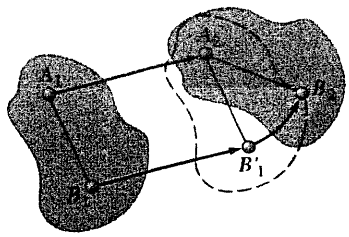
15.5 General Plane Motion

a general plane motion can always be considered as the sum of a translation and a rotation, as shown in the following examples



Plane motion = Translation with A + Rotation about A





in the general case of motion

A_1B_1 => A_2B_1' by translation

A_2B_1' => A_2B_2 by rotation

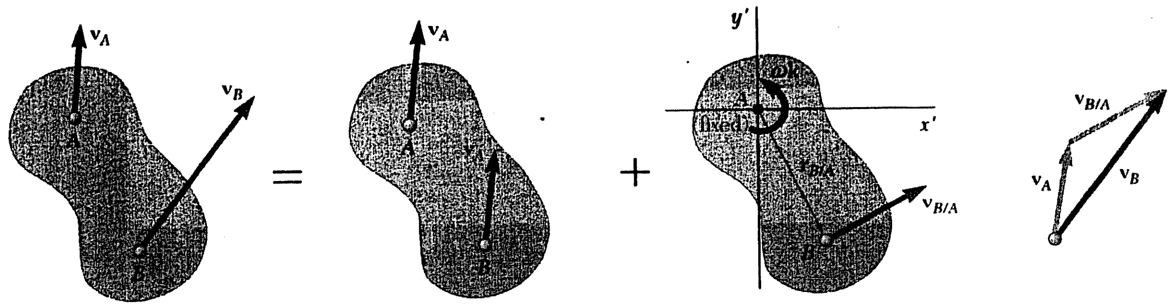
15.6 Absolute and Relative Velocity in Plane Motion

the absolute velocity $\boldsymbol{v}_B$ of a particle B is obtained

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

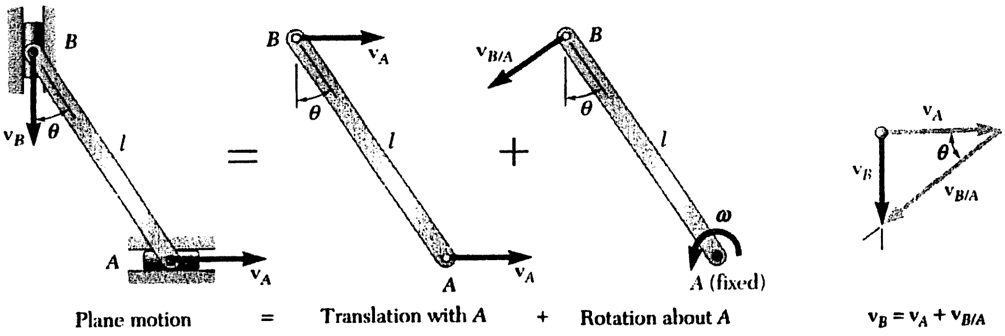
where $\boldsymbol{v}_{B/A} = \omega \boldsymbol{k} \times \boldsymbol{r}_{B/A}$ $v_{B/A} = r\,\omega$

then $\boldsymbol{v}_B = \boldsymbol{v}_A + \omega \boldsymbol{k} \times \boldsymbol{r}_{B/A}$



consider the rod AB, assuming that the velocity $\boldsymbol{v}_A$ is known, we want to find $\boldsymbol{v}_B$ and ω

take the reference point at A



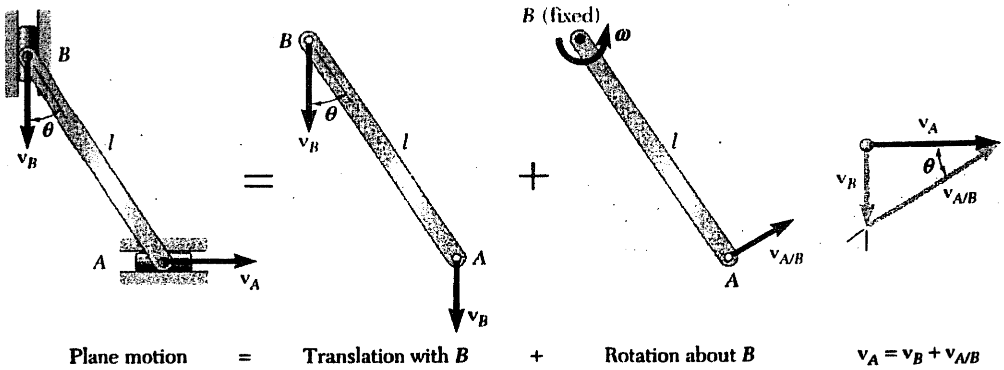
$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

the directions of $\boldsymbol{v}_A$, $\boldsymbol{v}_B$ and $\boldsymbol{v}_{B/A}$ are all known

$$v_B = v_A \tan\theta$$

$$\omega = v_{B/A} / l = v_A / l \cos\theta \quad \text{(CCW)}$$

take the reference point at B



$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$$

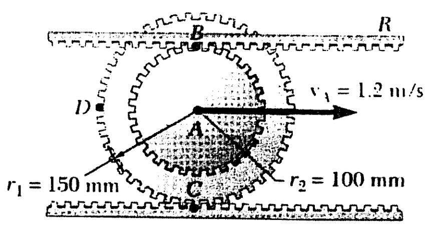
$$v_A = v_B \cot\theta \quad => \quad v_B = v_A \tan\theta \quad \text{(same)}$$

$$\omega = v_{A/B} / l = v_B / l \sin\theta = v_A \tan\theta / l \sin\theta$$

$$= v_A / l \cos\theta \quad \text{(same)} \quad \text{(CCW)}$$

the angular velocity ω of a rigid body in plane motion is independent of the reference point

Sample Problem 15.2



$\boldsymbol{v}_A = 1.2 \text{ m/s} \rightarrow$

determine ω, $\boldsymbol{v}_R$, and $\boldsymbol{v}_D$

after one revolution of the gear

$$x_A = 2\pi r_1 \qquad \theta = 2\pi$$

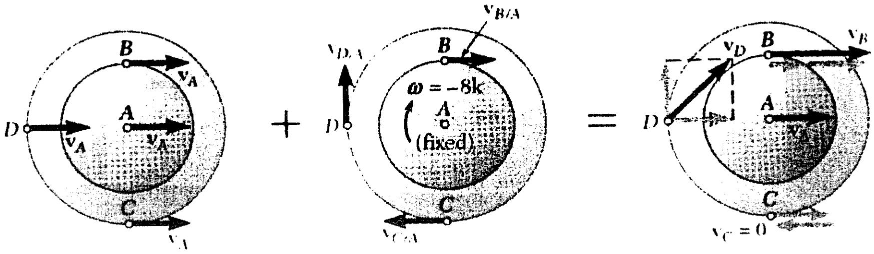
then $\quad x_A / 2\pi r_1 = -\theta / 2\pi$

or $\quad x_A = -r_1 \theta$

and $\quad v_A = -r_1 \dot{\theta} = -r_1 \omega$

$$1.2 = -0.15\,\omega$$

$$\omega = -8 \text{ rad/s} \qquad \text{or} \qquad \boldsymbol{\omega} = -8\,\boldsymbol{k} \text{ rad/s}$$



Translation + Rotation = Rolling Motion

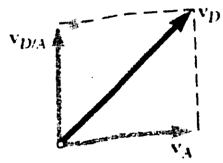
$$\boldsymbol{v}_R = \boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A} = \boldsymbol{v}_A + \omega\boldsymbol{k} \times \boldsymbol{r}_{B/A}$$

$$= 1.2\,\boldsymbol{i} + (-8\,\boldsymbol{k}) \times (0.1\,\boldsymbol{j}) = 2\,\boldsymbol{i} = 2 \text{ m/s} \rightarrow$$

$$\boldsymbol{v}_D = \boldsymbol{v}_A + \boldsymbol{v}_{D/A}$$

$$= 1.2\,\boldsymbol{i} + (-8\,\boldsymbol{k}) \times (-1.5\,\boldsymbol{i}) = 1.2\,\boldsymbol{i} + 1.2\,\boldsymbol{j}$$

$$= 1.697 \text{ m/s} \nearrow 45^{\circ}$$

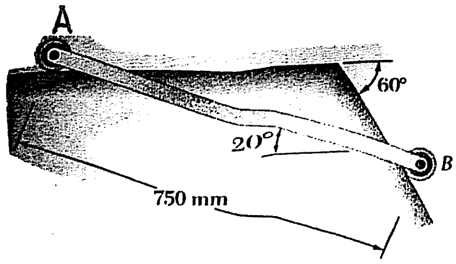


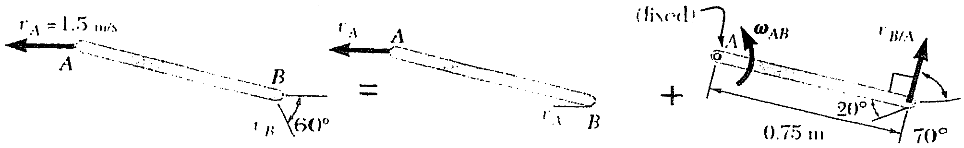
Sample Problem 15.3

small wheels have been attached to rod AB

$\boldsymbol{v}_A = 1.5$ m/s $\leftarrow$

determine $\boldsymbol{v}_B$ and ω



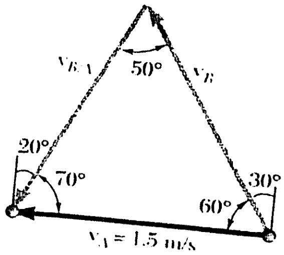


$$v_B = v_A + v_{B/A}$$

$$v_B \nwarrow 60^\circ = 1.5 \leftarrow + \; v_{B/A} \nearrow 70^\circ$$

law of sines

$$\frac{v_B}{\sin 70^\circ} = \frac{v_{B/A}}{\sin 70^\circ} = \frac{1.5}{\sin 70^\circ}$$



then $\quad v_B = 1.84 \text{ m/s} \nwarrow 60^\circ$

$$v_{B/A} = 1.696 \text{ m/s} \nearrow 70^\circ$$

but $\quad v_{B/A} = AB\,\omega$

$$1.696 = 0.75\,\omega$$

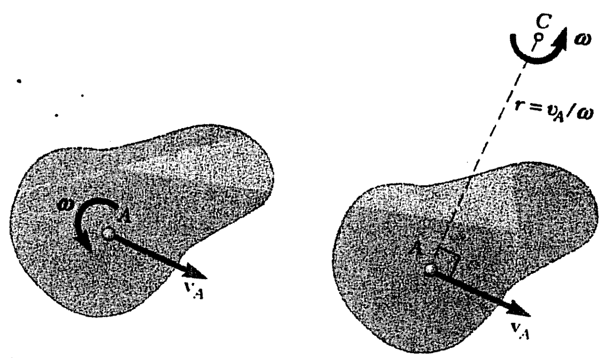
$$\omega = 2.261 \text{ rad/s counterclockwise}$$

15.7 Instantaneous Center of Rotation in Plane Motion

consider the general plane motion in slab, for a given time, the velocities of the various particles of the slab are the same if the slab were rotating about a certain axis perpendicular to the plane of slab is called ***instantaneous axis of rotation*** which intersect the slab at a point C, is called ***instantaneous center of rotation***

$$\therefore \quad v_B = v_A + \omega k \times r_{B/A}$$

a general plane motion may be described by a translation and rotation, it may be described by v_A and ω which rotates about C



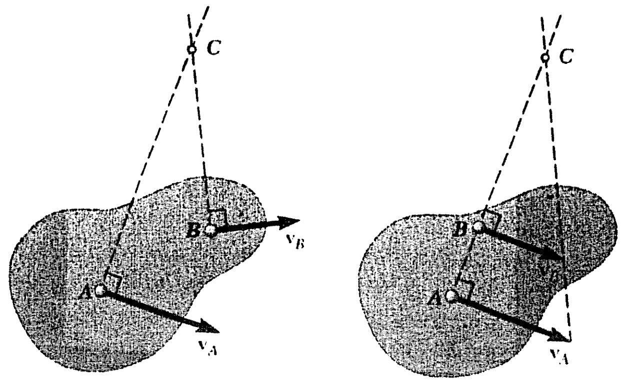
if v_A and ω are given, these velocities could be obtained by letting the slab rotate with the angular velocity ω about a point C located on the perpendicular to v_A at distance $r = v_A / \omega$

$\boldsymbol{v}_A \perp \boldsymbol{AC}$ $\qquad r = AC = v_A / \omega$

the slab seems to rotate about the instantaneous center C

the position of the instantaneous center can be defined in two other ways

1. intersection of two lines perpendicular to $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$
2. if $\boldsymbol{v}_A \mathbin{//} \boldsymbol{v}_B$, and $\boldsymbol{AB} \perp \boldsymbol{v}_A$, C can be found by intersecting the line AB with the line joining the extremities of the vectors $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$



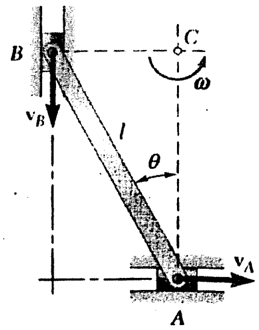
the instantaneous center C may be located either on or outside the slab

e.g. consider the rod AB

C can easily to locate (C in not inside AB)

$$\omega = v_A / AC = v_B / BC$$

$$v_B = v_A \, BC / AC = v_A \tan\theta$$



if C located on slab (say C'), then $v_{C'} = 0$ at this instant t (only at this given instant)

since the instantaneous center will move with time

$\therefore\ v_{C'} = 0$ at t $\quad$ but $\quad v_{C'} \neq 0$ at $t + \Delta t$

thus $\boldsymbol{a}_{C'} \neq 0$

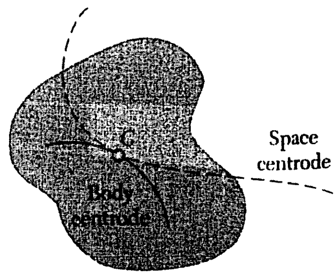
but the acceleration at this point cannot be determined directly

$$\boldsymbol{a} = \boldsymbol{a}_t + \boldsymbol{a}_n = r\,\alpha\,\boldsymbol{i}_t - v^2 / \rho\,\boldsymbol{i}_n$$

$$= 0\,\boldsymbol{i}_t - 0/0\,\boldsymbol{i}_n \quad \text{(no meaning)}$$

space centroid : instantaneous center describes one curve in space

body centroid : the locus of the position of instantaneous center on the body

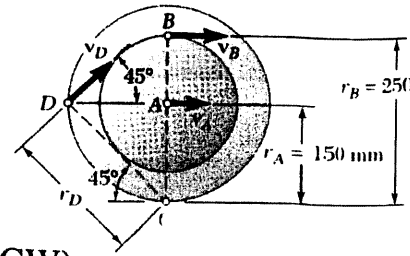


two curves are tangent at point C at instant body centroid curve rolls on the space centroid curve during motion of the body

Sample Problem 15.4

$\boldsymbol{v}_A = 1.2$ m/s $\rightarrow$

determine ω, $\boldsymbol{v}_R$, and $\boldsymbol{v}_D$



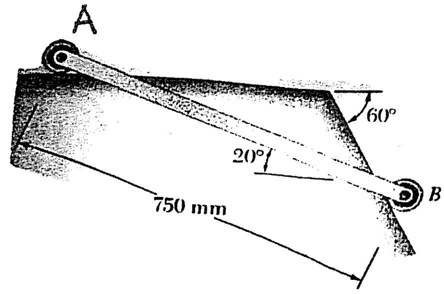
point C is the instantaneous center

$\omega = v_A / AC = 1.2 / 0.15 = 8$ rad/s (CW)

$v_R = v_B = r_B\,\omega = 0.25 \times 8 = 2$ m/s $\rightarrow$

$v_D = r_D\,\omega = 0.212 \times 8 = 1.697$ m/s $\angle 45^o$

Sample Problem 15.5



same problem as 15.3

from the given geometry,

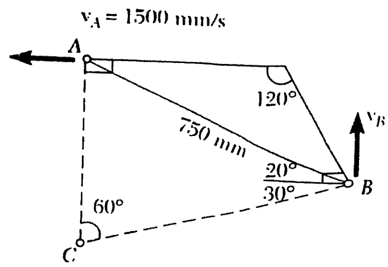
$\angle ACB = 60^o$

$\angle ABC = 20^o + 30^o = 50^o$

then $\angle BAC = 70^o$

$$\frac{AC}{\sin 50^o} = \frac{750}{\sin 60^o} = \frac{BC}{\sin 70^o}$$

$AC = 663.4$ mm $\qquad BC = .813.8$ mm



$\boldsymbol{v}_A = AC\,\omega$

$\omega = v_A / AC = 1500 / 663.4 = 2.26$ rad/s

$v_B = BC\,\omega \quad 813.8 \times 2.26 = 1839$ mm/s

$v_B = 1.84$ m/s $\nwarrow 60^o$

15.8 Absolute and Relative Acceleration in Plane Motion

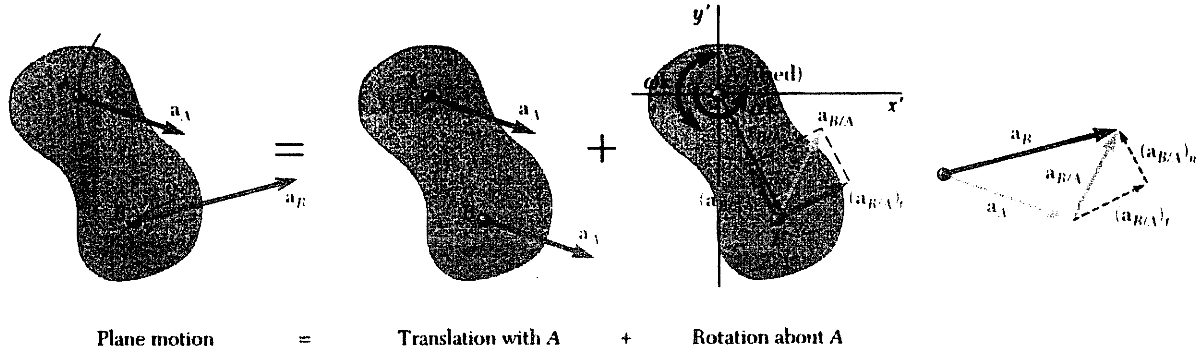
$$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$$

$$\boldsymbol{a}_{B/A} = (\boldsymbol{a}_{B/A})_t + (\boldsymbol{a}_{B/A})_n$$

$$(\boldsymbol{a}_{B/A})_t = \alpha\boldsymbol{k} \times \boldsymbol{r}_{B/A} \qquad (a_{B/A})_t = \alpha\, r$$

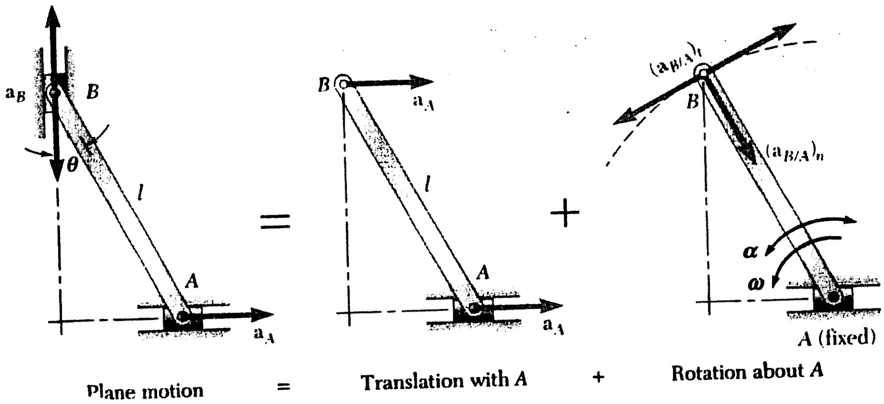
$$(\boldsymbol{a}_{B/A})_n = -\,\omega^2\, \boldsymbol{r}_{B/A} \qquad (a_{B/A})_n = r\,\omega^2$$

thus $$\boldsymbol{a}_B = \boldsymbol{a}_A + \alpha\boldsymbol{k} \times \boldsymbol{r}_{B/A} - \omega^2\, \boldsymbol{r}_{B/A}$$



e.g. consider the rod AB, with l, θ, $\boldsymbol{v}_A$ and $\boldsymbol{a}_A$ given

how to determine α_{AB} and $\boldsymbol{a}_B$



l, θ, $\boldsymbol{v}_A \rightarrow \boldsymbol{v}_B$ and ω_{AB}
(method of I. C.)

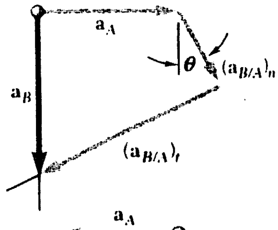
ω_{AB} and $\boldsymbol{r}_{B/A} \rightarrow (\boldsymbol{a}_{B/A})_n$

$\boldsymbol{a}_A$, $(\boldsymbol{a}_{B/A})_n$ and the directions of $\boldsymbol{a}_B$, $(\boldsymbol{a}_{B/A})_t$
$\rightarrow$ magnitudes of $\boldsymbol{a}_B$, $(\boldsymbol{a}_{B/A})_t$ by vector polygon

$(\boldsymbol{a}_{B/A})_t$ and $l \rightarrow \alpha_{AB}$

in the vector polygon of accelerations

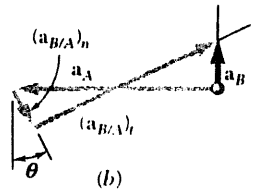
$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{a}_{B/A})_t + (\boldsymbol{a}_{B/A})_n$$

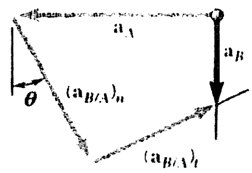


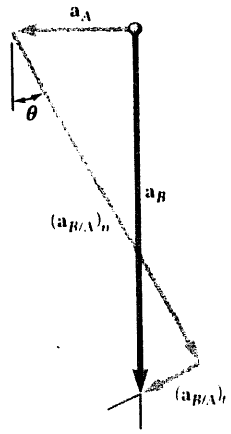
x-component $\quad 0 = a_A + l\,\omega^2 \sin\theta - l\,a\cos\theta$

y-component $\quad -a_B = -l\,\omega^2\cos\theta - l\,a\sin\theta$

two unknowns a_B and a can be solved



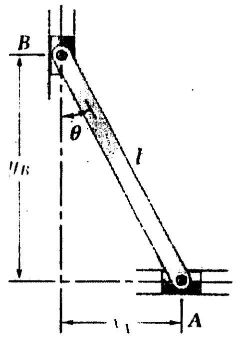




other three cases of vector polygons can be solved in the similarly way

15.9 Analysis of Plane Motion in Terms of a Parameter

consider the same example in section 15.8



$$x_A = l\sin\theta \qquad y_B = l\cos\theta$$

$$v_A = \dot{x}_A = -l\cos\theta\,\dot{\theta}$$

$$a_A = \dot{v}_A = -l\,\dot{\theta}^2\sin\theta + l\,\ddot{\theta}\cos\theta$$

$$v_B = \dot{y}_B = -l\sin\theta\,\dot{\theta}$$

$$a_B = \dot{v}_B = -l\,\dot{\theta}^2\cos\theta - l\,\ddot{\theta}\sin\theta$$

$$\because \quad \dot{\theta} = \omega \qquad \ddot{\theta} = a$$

then $\quad v_A = -l\cos\theta\,\omega \qquad v_B = -l\sin\theta\,\omega$

and $\quad \omega = v_A / l\cos\theta$ and $v_B = v_A\tan\theta$ can be solved

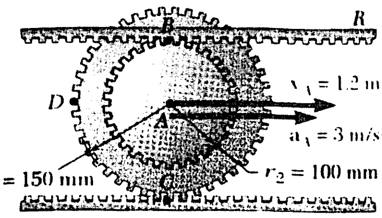
another two equations

$$a_A = -l\,\omega^2\sin\theta + l\,a\cos\theta$$

$$a_B = -l\,\omega^2\cos\theta - l\,a\sin\theta$$

two unknowns a_B and a can also be solved

Sample Problem 15.6



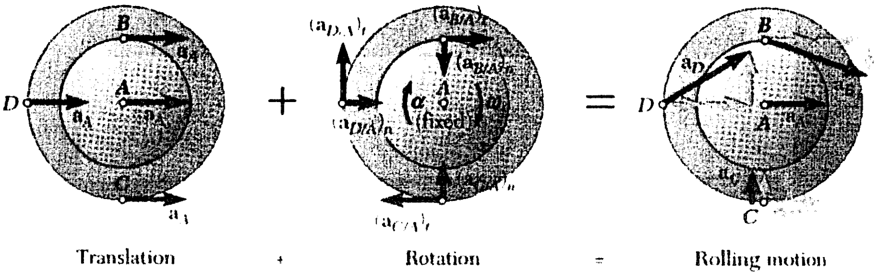
$\boldsymbol{v}_A = 1.2 \text{ m/s} \rightarrow \qquad \boldsymbol{a}_A = 3 \text{ m/s}^2 \rightarrow$

determine α, $\boldsymbol{a}_B$, $\boldsymbol{a}_C$ and $\boldsymbol{a}_D$

$$x_A = -r_1\,\theta$$

$$v_A = -r_1\,\omega \qquad \omega = -8 \text{ rad/s}$$

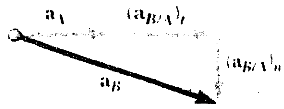
$$a_A = -r_1\,\alpha \qquad \alpha = -20 \text{ rad/s}^2$$



$$\boldsymbol{a}_B = \boldsymbol{a}_A + (\boldsymbol{a}_{B/A})_t + (\boldsymbol{a}_{B/A})_n$$

$$= 3\,\boldsymbol{i} - 20\,\boldsymbol{k} \times 0.1\,\boldsymbol{j} - 8^2 \times 0.1\,\boldsymbol{j}$$

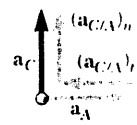
$$= 5\,\boldsymbol{i} - 6.4\,\boldsymbol{j} \;\; (\text{m/s}^2)$$



$$\boldsymbol{a}_C = \boldsymbol{a}_A + (\boldsymbol{a}_{C/A})_t + (\boldsymbol{a}_{C/A})_n$$

$$= 3\,\boldsymbol{i} - 20\,\boldsymbol{k} \times (-0.15\,\boldsymbol{j}) - 8^2 \times 0.15\,\boldsymbol{j}$$

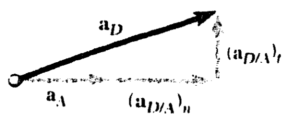
$$= -9.6\,\boldsymbol{j} \;\; (\text{m/s}^2)$$



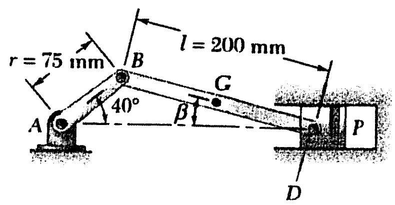
$$\boldsymbol{a}_D = \boldsymbol{a}_A + (\boldsymbol{a}_{D/A})_t + (\boldsymbol{a}_{D/A})_n$$

$$= 3\,\boldsymbol{i} - 20\,\boldsymbol{k} \times (-0.15\,\boldsymbol{i}) - 8^2 \times (-0.15\,\boldsymbol{j})$$

$$= 12.6\,\boldsymbol{i} - 3\,\boldsymbol{j} \;\; (\text{m/s}^2)$$



Sample Problem 15.7



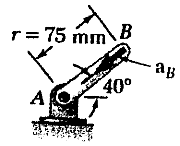
$\omega_{AB} = 2000 \text{ rpm} = 209.4 \text{ rad/s clockwise}$

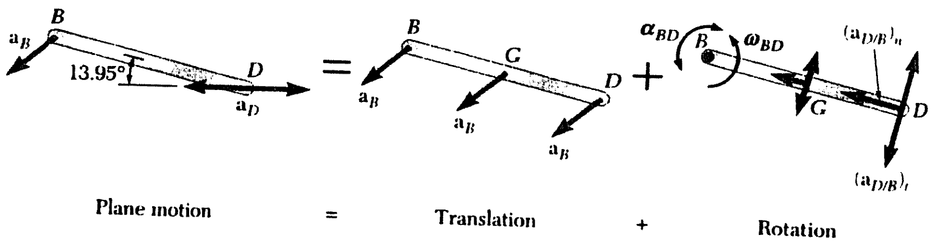
$\omega_{BD} = 62.0 \text{ rad/s counterclockwise}$

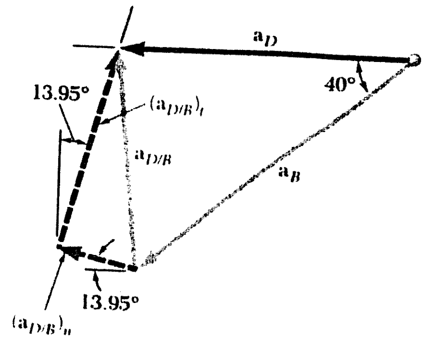
$\beta = 13.95°$

determine α_{BD} and $\boldsymbol{a}_D$

$$\boldsymbol{a}_B = \omega^2 AB = 209.4^2 \times 0.075 = 3289 \text{ m/s}^2 \angle 40°$$







$\boldsymbol{a}_D = \boldsymbol{a}_B + (\boldsymbol{a}_{D/B})_t + (\boldsymbol{a}_{D/B})_n$

$\boldsymbol{a}_D = a_D \longleftrightarrow$

$(\boldsymbol{a}_{D/B})_t = 0.2\, \alpha_{BD}$ ∠or∠ 76.1°

$(\boldsymbol{a}_{D/B})_n = 62^2 \times 0.2 = 769$ ◺ 13.95°

$a_D \longleftrightarrow = 3289 \angle 40^\circ + 0.2\, \alpha_{BD} \angle 76.1^\circ + 769$ ◺ 13.95°

x-comp

$$-a_D = -3289 \cos 40^\circ - 769 \cos 13.95^\circ + 0.2\, \alpha_{BD} \sin 76.1^\circ$$

y-comp

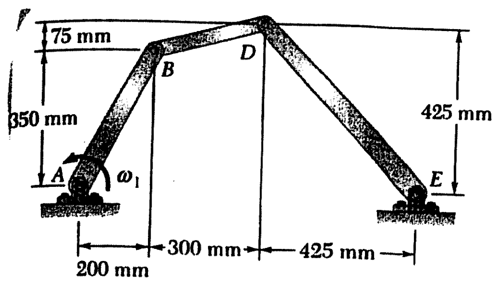
$$0 = -3289 \sin 40^\circ + 769 \sin 13.95^\circ + 0.2\, \alpha_{BD} \cos 76.1^\circ$$

solving the equations and obtained

$\alpha_{BD} = 9940 \text{ rad/s}^2$ counterclockwise

$\boldsymbol{a}_D = 2787 \text{ m/s}^2 \leftarrow$

Sample Problem 15.8

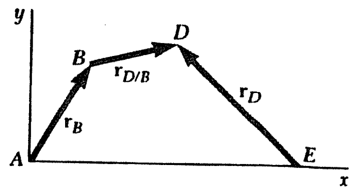


linkage *ABDE*

$\omega_1 = 20\,\boldsymbol{k}$ rad/s = constant (CCW)

determine ω_{BD}, ω_{DE}, α_{BD} and α_{DE}

$\boldsymbol{\omega}_{BD} = \omega_{BD}\,\boldsymbol{k} \qquad \boldsymbol{\omega}_{DE} = \omega_{DE}\,\boldsymbol{k}$



$\boldsymbol{r}_{B/A} = 200\,\boldsymbol{i} + 350\,\boldsymbol{j}$ (mm)

$\boldsymbol{r}_{D/B} = 300\,\boldsymbol{i} + 75\,\boldsymbol{j}$ (mm)

$\boldsymbol{r}_{D/E} = -425\,\boldsymbol{i} + 425\,\boldsymbol{j}$ (mm)

$\boldsymbol{\alpha}_{AB} = 0 \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\,\boldsymbol{k} \qquad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\,\boldsymbol{k}$

$$\boldsymbol{v}_D = \boldsymbol{v}_E + \boldsymbol{v}_{D/E} = \boldsymbol{v}_B + \boldsymbol{v}_{D/B}$$

$$\omega_{DE}\,\boldsymbol{k} \times \boldsymbol{r}_{D/E} = \omega_{AB}\,\boldsymbol{k} \times \boldsymbol{r}_{B/A} + \omega_{DB}\,\boldsymbol{k} \times \boldsymbol{r}_{D/B}$$

with ω_{AB} = 20 rad/s, two unknowns ω_{BD} and ω_{DE} can be solved by the above vector equation (two scalar equations)

ω_{BD} = - 29.33 rad/s (clockwise)

ω_{DE} = 11.29 rad/s (counterclockwise)

$$\boldsymbol{a}_D = \boldsymbol{a}_E + \boldsymbol{a}_{D/E} = \boldsymbol{a}_B + \boldsymbol{a}_{D/B}$$

$$\alpha_{DE}\,\boldsymbol{k} \times \boldsymbol{r}_{D/E} - \omega_{DE}^2\,\boldsymbol{r}_{D/E} = -\,\omega_{AB}^2\,\boldsymbol{r}_{B/A} + \alpha_{BD}\,\boldsymbol{k} \times \boldsymbol{r}_{D/B} - \omega_{BD}^2\,\boldsymbol{r}_{D/B}$$

two unknowns α_{BD} and α_{DE} can be solved by the above vector equation (two scalar equations)

α_{BD} = - 645 rad/s^2 (clockwise)

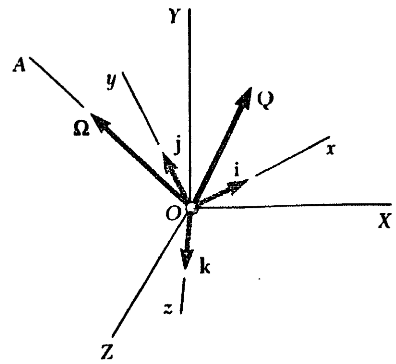
α_{DE} = 809 rad/s^2 (counterclockwise)

15.10 Rate of Change of a Vector with Respect to a Rotating Frame

OXYZ : frame of reference

Oxyz : rotating frame with rotating axis *OA*

consider a function $\boldsymbol{Q} = \boldsymbol{Q}(t)$



define $(\dot{\boldsymbol{Q}})_{XYZ}$: rate of change of $\boldsymbol{Q}$ w.r.t. *OXYZ*

$(\dot{\boldsymbol{Q}})_{xyz}$: rate of change of $\boldsymbol{Q}$ w.r.t. *Oxyz*

$$\boldsymbol{Q} = Q_x\,\boldsymbol{i} + Q_y\,\boldsymbol{j} + Q_z\,\boldsymbol{k}$$

$$(\dot{\boldsymbol{Q}})_{xyz} = \dot{Q}_x\,\boldsymbol{i} + \dot{Q}_y\,\boldsymbol{j} + \dot{Q}_z\,\boldsymbol{k}$$

and $$(\dot{\boldsymbol{Q}})_{XYZ} = \dot{Q}_x\,\boldsymbol{i} + \dot{Q}_y\,\boldsymbol{j} + \dot{Q}_z\,\boldsymbol{k} + Q_x\frac{d\boldsymbol{i}}{dt} + Q_y\frac{d\boldsymbol{j}}{dt} + Q_z\frac{d\boldsymbol{k}}{dt}$$

$(\dot{\boldsymbol{Q}})_{xyz} = 0$ if $\boldsymbol{Q}$ is fixed on *Oxyz*

$(\dot{\boldsymbol{Q}})_{XYZ}$: the velocity of particle located at the tip of $\boldsymbol{Q}$ and belonging to a body rigidly attached to the frame *Oxyz*

if $\boldsymbol{Q}$ is fixed on *Oxyz*

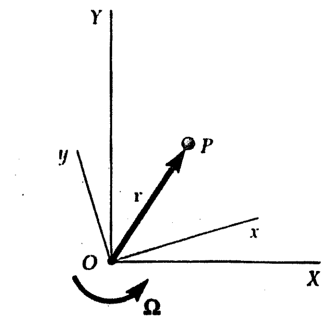
$$(\dot{\boldsymbol{Q}})_{XYZ} = Q_x \frac{di}{dt} + Q_y \frac{dj}{dt} + Q_z \frac{dk}{dt} = \boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{Q}$$

then in general

$$(\dot{\boldsymbol{Q}})_{XYZ} = (\dot{\boldsymbol{Q}})_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{Q}$$

the last term $\boldsymbol{\Omega} \times \boldsymbol{Q}$ is induced by rotation of *Oxyz*

15.11 Plane Motion of a Particle Relative to a Rotating Frame, Coriolis Acceleration



consider two frames of reference on a plane

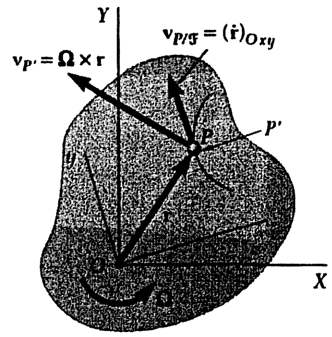
OXY : fixed frame

Oxy : rotating frame

let $\boldsymbol{v}_P$ = absolute velocity of *P*

then

$$\boldsymbol{v}_P = (\dot{\boldsymbol{r}})_{OXY} = (\dot{\boldsymbol{r}})_{Oxy} + \Omega \times \boldsymbol{r}$$
$$= \boldsymbol{v}_{P/F} + \boldsymbol{v}_{P'}$$



$\boldsymbol{v}_{P'}$: velocity of point *P'* on the moving frame which coincide with *P* at the instant

$\boldsymbol{v}_{P/F}$: velocity of *P* relative to *Oxy*

let $\boldsymbol{a}_P$ be the absolute acceleration

$$\boldsymbol{a}_P = \dot{\boldsymbol{v}}_P = \dot{\Omega} \times \boldsymbol{r} + \Omega \times \dot{\boldsymbol{r}} + \frac{d}{dt}[(\dot{\boldsymbol{r}})_{Oxy}]$$

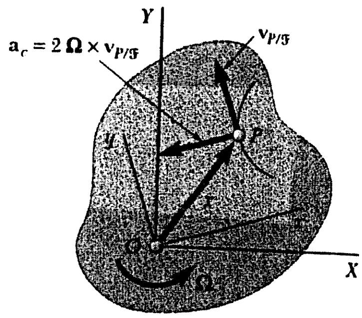
$$\frac{d}{dt}[(\dot{\boldsymbol{r}})_{Oxy}] = [(\dot{\boldsymbol{r}})_{Oxy}]^{\bullet}_{OXY} = (\ddot{\boldsymbol{r}})_{Oxy} + \Omega \times (\dot{\boldsymbol{r}})_{Oxy}$$

$$\dot{\boldsymbol{r}} = (\dot{\boldsymbol{r}})_{OXY} = (\dot{\boldsymbol{r}})_{Oxy} + \Omega \times \boldsymbol{r}$$

then $\boldsymbol{a}_P = \dot{\Omega} \times \boldsymbol{r} + \Omega \times (\Omega \times \boldsymbol{r}) + 2\,\Omega \times (\dot{\boldsymbol{r}})_{Oxy} + (\ddot{\boldsymbol{r}})_{Oxy}$

$= (\boldsymbol{a}_{P'})_t + (\boldsymbol{a}_{P'})_n + \boldsymbol{a}_C + \boldsymbol{a}_{P/F}$

$\boldsymbol{a}_P = \boldsymbol{a}_{P'} + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C$



$\boldsymbol{a}_{P'}$: acceleration of point P' of the moving frame coincide with P at instant

$\boldsymbol{a}_{P/F}$: acceleration of P relative to moving frame

$\boldsymbol{a}_C$: $\boldsymbol{a}_C = 2\,\Omega \times (\dot{\boldsymbol{r}})_{Oxy} = 2\,\Omega \times (\boldsymbol{v}_{P/F})$ is the complementary or Coriolis acceleration

$\boldsymbol{a}_C \perp \Omega$ and $\boldsymbol{v}_{P/F}$, $\boldsymbol{a}_C = 0$ when Ω or $\boldsymbol{v}_{P/F} = 0$

consider a collar P which is made to slide along a rotating rod OB

time from t to $t + \Delta t$

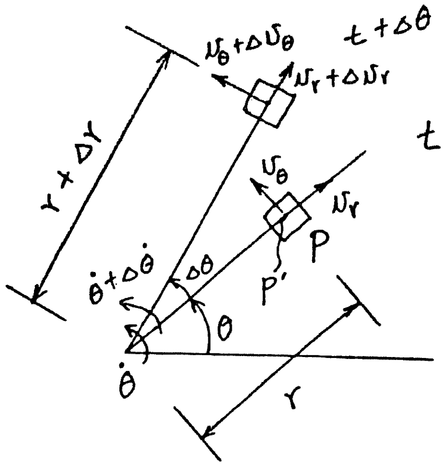
$r \rightarrow r + \Delta r$

$\theta \rightarrow \theta + \Delta\theta$

$\dot{\theta} \rightarrow \dot{\theta} + \Delta\dot{\theta}$

$v_r \rightarrow v_r + \Delta v_r$

$v_\theta \rightarrow v_\theta + \Delta v_\theta$



for the velocity vector polygon

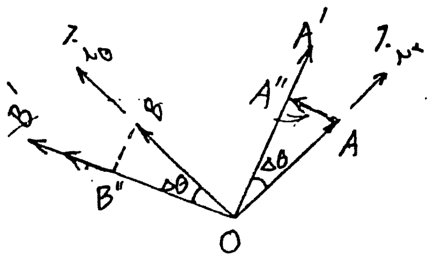
$OA = v_r = \dot{r}$

$OB = v_\theta = r\,\dot{\theta}$

$OA' = v_r + \Delta v_r$

$OB' = v_\theta + \Delta v_\theta = (r + \Delta r)(\dot{\theta} + \Delta\dot{\theta})$

$= r\,\dot{\theta} + \Delta r\,\dot{\theta} + r\,\Delta\dot{\theta} + \Delta r\,\Delta\dot{\theta}$



at time t $\quad \boldsymbol{v} = \boldsymbol{OA} + \boldsymbol{OB}$

at time $t + \Delta t$ $\quad \boldsymbol{v} + \Delta\boldsymbol{v} = \boldsymbol{OA'} + \boldsymbol{OB'}$

then $\quad \Delta\boldsymbol{v} = (\boldsymbol{OA'} + \boldsymbol{OA}) + (\boldsymbol{OB'} + \boldsymbol{OB})$

$= \boldsymbol{AA''} + \boldsymbol{A''A'} + \boldsymbol{BB''} + \boldsymbol{B''B'}$

$$AA" = OA\,\Delta\theta\,\boldsymbol{i}_\theta = \dot{r}\,\Delta\theta\,\boldsymbol{i}_\theta$$

$$A"A' = \Delta v_r\,\boldsymbol{i}_r$$

$$BB" = -OB\,\Delta\theta\,\boldsymbol{i}_\theta = -r\,\dot{\theta}\,\Delta\theta\,\boldsymbol{i}_r$$

$$B"B' = (\Delta r\,\dot{\theta} + r\,\Delta\dot{\theta})\,\boldsymbol{i}_\theta$$

$$\Delta\boldsymbol{v} = (\Delta v_r - r\,\dot{\theta}\,\Delta\theta)\,\boldsymbol{i}_r + (\dot{r}\,\Delta\theta + \Delta r\,\dot{\theta} + r\,\Delta\dot{\theta})\,\boldsymbol{i}_\theta$$

take $\Delta t \rightarrow 0$

$$\boldsymbol{a}_P = \lim\frac{\Delta v}{\Delta t} = \left(\frac{dv_r}{dt} - r\,\dot{\theta}\frac{d\theta}{dt}\right)\boldsymbol{i}_r + \left(\dot{r}\frac{d\theta}{dt} + \frac{dr}{dt}\dot{\theta} + r\frac{d\dot{\theta}}{dt}\right)\boldsymbol{i}_\theta$$

$$= (\ddot{r} - r\,\dot{\theta}^2)\,\boldsymbol{i}_r + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\boldsymbol{i}_\theta$$

$$= (-r\,\dot{\theta}^2\,\boldsymbol{i}_r + r\,\ddot{\theta}\,\boldsymbol{i}_\theta) + \ddot{r}\,\boldsymbol{i}_r + 2\,\dot{r}\,\dot{\theta}\,\boldsymbol{i}_\theta$$

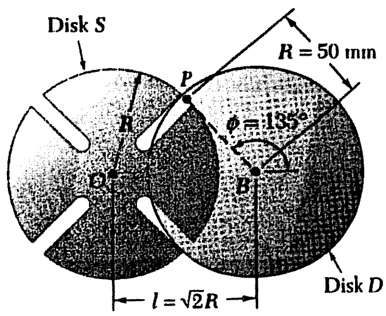
$$= \qquad \boldsymbol{a}_{P'} \qquad + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C$$

$\boldsymbol{a}_C = 2\,\dot{r}\,\dot{\theta}\,\boldsymbol{i}_\theta$ inculde two components :

1. the direction of relative velocity changed (***AA"***)

2. the distance from O is changed, thus the tangential velcoity component changed (***B"B'***)

Sample Problem 15.9



ω_D = 10 rad/s counterclockwise

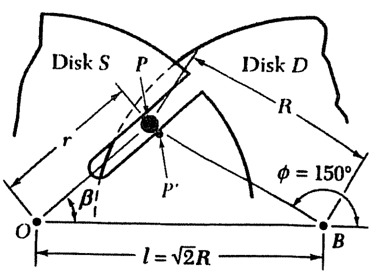
$\theta = 135^\circ \quad R = 50$ mm

when $\phi = 150^\circ$, determine ω_S and $\boldsymbol{v}_{P/S}$

$$r^2 = R^2 + l^2 - 2\,R\,l\cos 30^\circ = 0.551\,R^2$$

$$r = 37.1 \text{ mm}$$

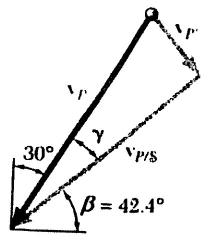
$$\frac{50}{\sin\beta} = \frac{37.1}{\sin 30^\circ} \quad \Rightarrow \quad \beta = 42.4^\circ$$



∵ P is attached on D

$\therefore \quad \boldsymbol{v}_P = R\,\omega_D = 50 \times 10 = 500$ mm/s ↙ 60°

from dish *S*



$$\boldsymbol{v}_P = \boldsymbol{v}_{P'} + \boldsymbol{v}_{P/S}$$

$$\gamma = 90^\circ - 30^\circ - 42.4^\circ = 17.6^\circ$$

$\boldsymbol{v}_{P'} \perp \boldsymbol{v}_{P/S} \qquad \boldsymbol{v}_{P'} = v_P \sin\gamma = 151.2$ mm/s |↘ 42.4°

$\boldsymbol{v}_{P/S} = v_P \cos\gamma = 477$ mm/s ↙ 42.4°

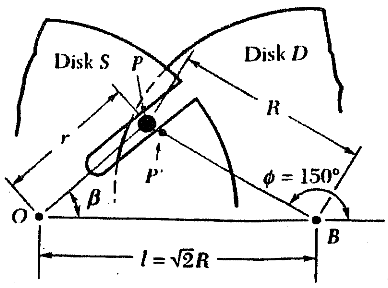
and $\qquad v_P = r\,\omega_S$

$\omega_S = 4.08$ rad/s clockwise

Sample Problem 15.10

same condition as state in 15.9

determine $\boldsymbol{a}_S$



from sample problem 15.9

$\omega_S = 4.08$ rad/s clockwise

$\beta = 42.4^\circ \quad \boldsymbol{v}_{P/S} = 477$ mm/s ↙ 42.4°

$$\because \qquad \boldsymbol{a}_P = \boldsymbol{a}_{P'} + \boldsymbol{a}_{P/S} + \boldsymbol{a}_C$$

$\boldsymbol{a}_P = (\boldsymbol{a}_P)_n = -R\,\omega_D^2\,\boldsymbol{i}_n = 5000$ mm/s² ↘ 30°

$[\because \omega_D = \text{constant}, \ (\boldsymbol{a}_P)_t = 0]$

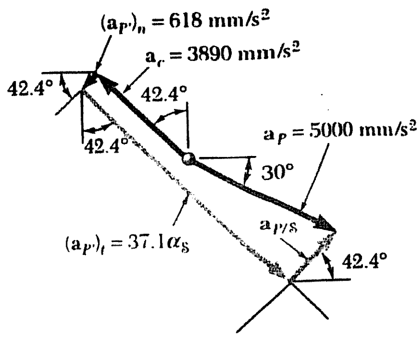
$\boldsymbol{a}_{P'} = (\boldsymbol{a}_{P'})_t + (\boldsymbol{a}_{P'})_n = r\,\alpha_S\,\boldsymbol{i}_t - r\,\omega_S^2\,\boldsymbol{i}_n$

$= 37.1\,\alpha_S\,\boldsymbol{i}_t - 618\,\boldsymbol{i}_n$

$\boldsymbol{a}_{P/S} = a_{P/S}$ ↗ or ↙ 42.4°

$\boldsymbol{a}_C = 2\,\omega_S \times \boldsymbol{v}_{P/S} = 2 \times 4.08 \times 447$

$= 3890$ mm/s ↖| 42.4°



from the vector polygon diagram

$(a_{P'})_t = a_C + a_P \cos 17.6^o$

$37.1\ a_S = 3890 + 5000 \cos 17.6^o$

$a_S = 233$ rad/s^2 clockwise

$a_{P/S} = (a_{P'})_n + a_P \sin 17.6^o = 768$ mm/s^2 $\nearrow$ 42.4^o

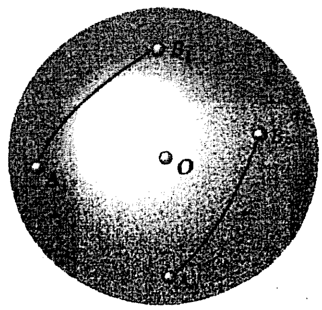
15.12 Motion about a Fixed Point

at instant, motion about a fixed point may be considered as a rotation of the body about an axis (called instantaneous axis of rotation) through O

consider the body rotates about O

$(A_1B_1) \rightarrow (A_2B_2)$

but $A_1B_1 = A_2B_2$ (rigid body)



(a)

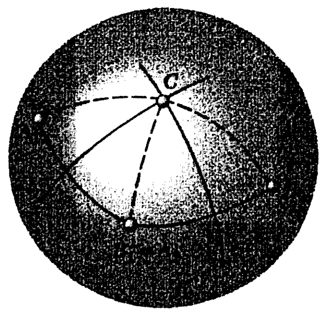
let A_2 at B_1, and

let C be the point of intersection of these two bisecting arcs, then

$A_1C = A_2C = B_2C$

$\because$ $A_1B_1 = A_2B_2$

$\therefore$ $\Delta A_1CB_1 = \Delta A_2CB_2$



denote $\theta = \angle A_1CB_1$

the motion of the sphere during Δt may be considered as a rotation through $\Delta\theta$ about OC

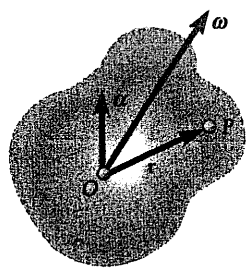
let $\Delta t \rightarrow 0$, OC is called the instantaneous axis of rotation

at this instant

$\boldsymbol{v} = d\boldsymbol{r} / dt = \omega \times \boldsymbol{r}$

$\boldsymbol{a} = \alpha \times \boldsymbol{r} + \omega \times (\omega \times \boldsymbol{r})$

where $\alpha = d\omega / dt$



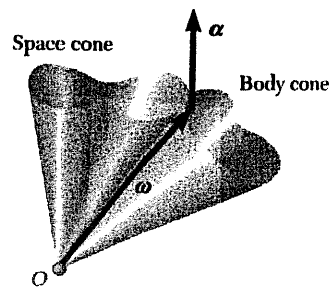
∵ ω is change in direction and magnitude with time

thus α is not directed along the instantaneous axis of rotation

α is tangent to the curve described in the space by the tip of vector ω

space cone : instantaneous axis of rotation describes on cone in space

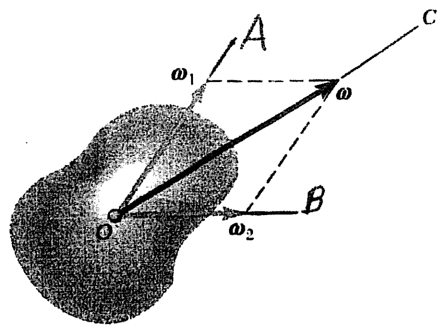
body cone : the locus of the position of instantaneous axis of rotation in the body



at any instant, two cones are tangent along the instantaneous axis or rotation, the body cone appears to roll on the space cone

the vector ω moves within the body as well as in space

consider a rigid body fixed at O, rotates simultaneously about OA and OB, may be considered equivalent a single rotation of $\vec{\omega}$



$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$$

let point P inside the body of position vector $\boldsymbol{r}$

denote $\quad \boldsymbol{v}_1$: velocity of P when the body rotates about OA

$\boldsymbol{v}_2$: velocity of P when the body rotates about OB

then $\quad \boldsymbol{v}_1 = \vec{\omega}_1 \times \boldsymbol{r} \qquad \boldsymbol{v}_2 = \vec{\omega}_2 \times \boldsymbol{r}$

and $\quad \boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 = (\vec{\omega}_1 + \vec{\omega}_2) \times \boldsymbol{r} = \vec{\omega} \times \boldsymbol{r}$

thus $\quad \vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2$

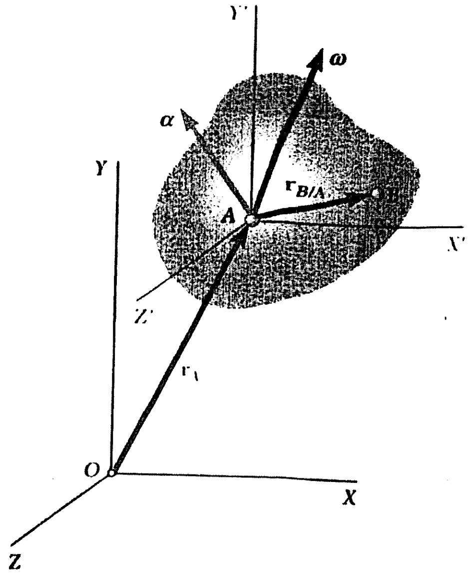
15.13 General Motion

general motion = translation + rotation about a fixed point

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}$$

$= \boldsymbol{v}_A + \omega \times \boldsymbol{r}_{B/A}$



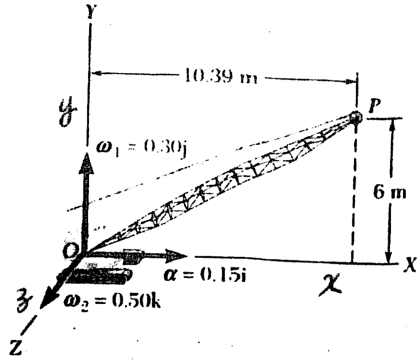
where ω is the angular velocity of the body at instant

$\boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}$

$= \boldsymbol{a}_A + \alpha \times \boldsymbol{r}_{B/A} + \omega \times (\omega \times \boldsymbol{r}_{B/A})$

where α is the angular acceleration of the body at instant

Sample Problem 15.11



$\omega_1 = 0.3$ rad/s

$\omega_2 = 0.5$ rad/s relative to the cab ($Oxyz$)

determine ω_{OP}, α_{OP}, $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$

$\omega_{OP} = \omega_1 + \omega_2 = 0.3\boldsymbol{j} + 0.5\boldsymbol{k}$ (rad/s)

$\alpha_{OP} = [\dot{\omega}_{OP}]_{OXYZ} = [\dot{\omega}_1]_{OXYZ} + [\dot{\omega}_2]_{OXYZ}$

$= 0 + (\dot{\omega}_2)_{Oxyz} + \omega_1 \times \omega_2$

$= 0.3\boldsymbol{j} \times 0.5\boldsymbol{k} = 0.15\boldsymbol{i}$ (rad/s^2)

$\boldsymbol{v}_P = \omega_{OP} \times \boldsymbol{r}_P = (0.3\boldsymbol{j} + 0.5\boldsymbol{k}) \times (10.39\boldsymbol{i} + 6\boldsymbol{j})$

$= -3\boldsymbol{i} + 5.2\boldsymbol{j} - 3.12\boldsymbol{k}$ (m/s)

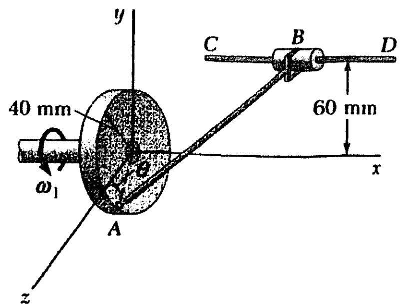
$\boldsymbol{a}_P = \alpha_{OP} \times \boldsymbol{r}_P + \omega_{OP} \times (\omega_{OP} \times \boldsymbol{r}_P)$

$= -3.54\boldsymbol{i} - 1.5\boldsymbol{j} + 1.8\boldsymbol{k}$ (m/s^2)

Sample Problem 15.12

$\omega_1 = 12\boldsymbol{i}$ rad/s CD on xy plane

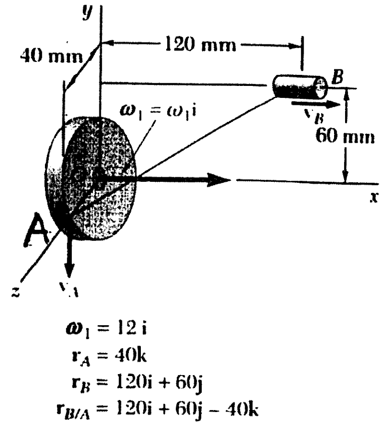
determine $\boldsymbol{v}_B$ and ω_{AB} for $\theta = 0$



$\boldsymbol{v}_A = \omega_1 \times \boldsymbol{r}_A = 12\boldsymbol{i} \times 40\boldsymbol{k} = -480\boldsymbol{j}$ (mm/s)

$\boldsymbol{v}_B = v_B\boldsymbol{i}$ $\omega_{AB} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$

$\boldsymbol{v}_B = \boldsymbol{v}_A + \omega \times \boldsymbol{r}_{B/A}$



$$v_B \boldsymbol{i} = -480\boldsymbol{j} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ 120 & 60 & -40 \end{vmatrix}$$

$$v_B = -40\,\omega_y - 60\,\omega_z \qquad (1)$$

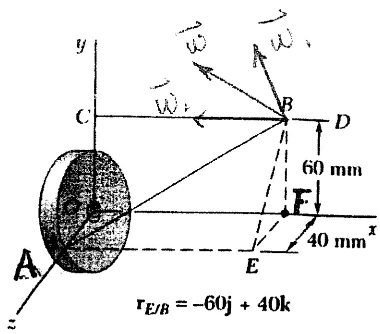
$$-480 = 40\,\omega_x + 120\,\omega_z \qquad (2)$$

$$0 = 60\,\omega_x - 120\,\omega_y \qquad (3)$$

3 equations for 4 unknowns

(1) x 6 + (2) x 3 + (3) x (-2)

$$6\,v_B = -1440 \qquad \boldsymbol{v}_B = -240\,\boldsymbol{i}$$



∵ *AB* rotates about *CD*, also rotates about an axis ⊥ plane containing *AB* and *CD*

$\omega_1 \perp$ ***BE*** and ***AB***   plane *ABE*

$\omega_2 \perp$ ***BE*** and ***BF***   plane *BEF*

i.e. $\omega_1 \parallel \boldsymbol{BC} \times \boldsymbol{BA}$

$\omega_2 \parallel \boldsymbol{BC}$

thus ω in on the plane containing vectors ($\boldsymbol{BC} \times \boldsymbol{BA}$) and $\boldsymbol{BC}$

i.e. $\omega \perp \boldsymbol{BC} \times (\boldsymbol{BC} \times \boldsymbol{BA})$

∵ $\boldsymbol{BE} \parallel \boldsymbol{BC} \times (\boldsymbol{BC} \times \boldsymbol{BA})$

thus $\omega \perp \boldsymbol{BE}$

or $\omega \cdot \boldsymbol{BE} = 0$

$$(\omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}) \cdot (-60\boldsymbol{j} + 40\boldsymbol{k}) = 0$$

$$-60\,\omega_y + 40\,\omega_z = 0 \qquad (4)$$

from equations (1) to (4), it is obtained

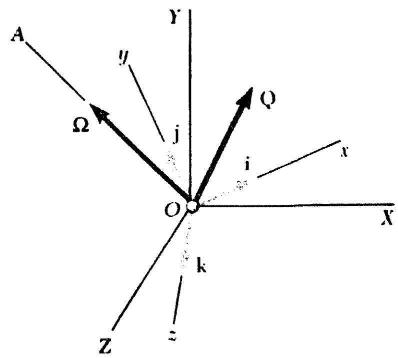
$$\omega = 3.69\boldsymbol{i} + 1.846\boldsymbol{j} + 2.77\,\boldsymbol{k} \quad \text{(rad/s)}$$

15.14 Three-Dimensional Motion of a Particle Relative to a Rotating Frame, Coriolis Acceleration

OXYZ : fixed coordinate

Oxyz : rotating coordinate

OA : instantaneous axis of rotation

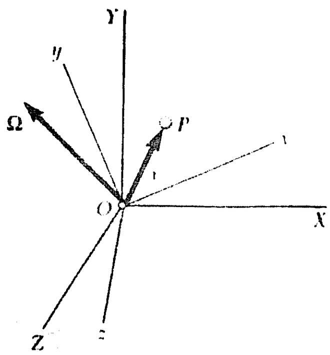


$$(\dot{\boldsymbol{Q}})_{OXYZ} = (\dot{\boldsymbol{Q}})_{Oxyz} + \Omega \times \boldsymbol{Q}$$

$$\boldsymbol{v}_P = (\dot{\boldsymbol{r}})_{OXYZ} = (\dot{\boldsymbol{r}})_{Oxyz} + \Omega \times \boldsymbol{r}$$

$$= \boldsymbol{v}_{P/F} + \boldsymbol{v}_{P'}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{P'} + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C$$



$$\boldsymbol{a}_{P'} = \dot{\Omega} \times \boldsymbol{r} + \Omega \times (\Omega \times \boldsymbol{r})$$

$$\boldsymbol{a}_{P/F} = (\ddot{\boldsymbol{r}})_{Oxyz}$$

$$\boldsymbol{a}_C = 2\,\Omega \times (\dot{\boldsymbol{r}})_{Oxyz}$$ Coriolis acceleration

$\boldsymbol{a}_C \perp \Omega$ and $\boldsymbol{v}_{P/F}$ but Ω not $\perp$ $\boldsymbol{v}_{P/F}$ in general

thus $a_C \neq 2\,\Omega\, \boldsymbol{v}_{P/F}$ in general

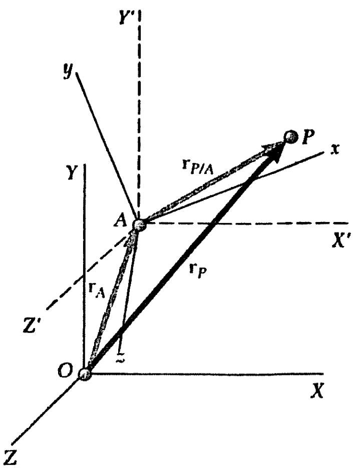
15.15 Frame of Reference in General Motion

OXYZ : fixed coordinate

AX'Y'Z' : translation coordinate

Axyz : rotating coordinate

Ω : angular velocity of *Axyz* at instant



$$\boldsymbol{r}_P = \boldsymbol{r}_A + \boldsymbol{r}_{P/A}$$

$$\boldsymbol{v}_P = \boldsymbol{v}_A + [\dot{\boldsymbol{r}}_{P/A}]_{AX'Y'Z'}$$

$$= \boldsymbol{v}_A + (\dot{\boldsymbol{r}}_{P/A})_{Axyz} + \Omega \times \boldsymbol{r}_{P/A}$$

$$\boldsymbol{a}_P = [\dot{\boldsymbol{v}}_P]_{OXYZ} = \dot{\boldsymbol{v}}_A + [\dot{\boldsymbol{v}}_{P/A}]_{AX'Y'Z'}$$

$$= \boldsymbol{a}_A + \dot{\Omega} \times \boldsymbol{r}_{P/A} + \Omega \times (\Omega \times \boldsymbol{r}_{P/A}) + (\ddot{\boldsymbol{r}}_{P/A})_{Axyz} + 2\,\Omega \times (\dot{\boldsymbol{r}}_{P/A})_{Axyz}$$

$$= \boldsymbol{a}_A + \qquad \boldsymbol{a}_{P'} \qquad + \boldsymbol{a}_{P/A} + \boldsymbol{a}_C$$

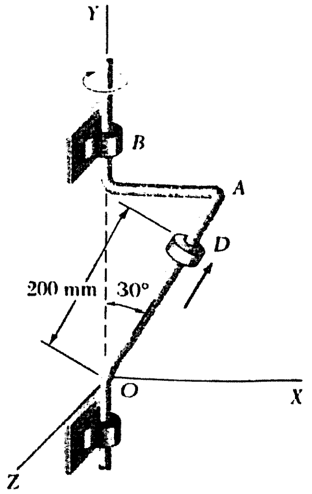
Sample Problem 15.13

$\Omega = -20\,\boldsymbol{j}$ rad/s $\qquad \dot{\Omega} = -200\,\boldsymbol{j}$ rad/s^2

$\boldsymbol{v}_{D/F} = 1.25$ m/s $\angle 60^\circ \qquad \boldsymbol{a}_{D/F} = 1.25$ m/s^2 $\angle 60^\circ$

$OD = 200$ mm

determine $\boldsymbol{v}_D$ and $\boldsymbol{a}_D$



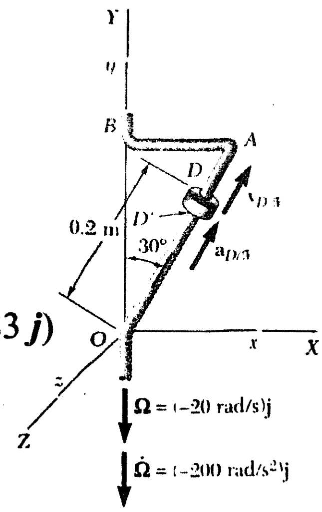
choose OAB as the rotating frame

$$\boldsymbol{r}_D = 0.2\,(\sin 30^\circ\,\boldsymbol{i} + \cos 30^\circ\,\boldsymbol{j})$$

$$= 0.1\,\boldsymbol{i} + 0.1732\,\boldsymbol{j}\ \text{(m)}$$

$$\boldsymbol{v}_{D/F} = 0.625\,\boldsymbol{i} + 1.083\,\boldsymbol{j}\ \text{(m/s)}$$

$$\boldsymbol{a}_{D/F} = 7.5\,\boldsymbol{i} + 12.99\,\boldsymbol{j}\ \text{(m/s}^2\text{)}$$



$$\boldsymbol{v}_D = \boldsymbol{v}_{D'} + \boldsymbol{v}_{D/F}$$

$$= -20\,\boldsymbol{j} \times (0.1\,\boldsymbol{i} + 0.1732\,\boldsymbol{j}) + (0.625\,\boldsymbol{i} + 1.083\,\boldsymbol{j})$$

$$= 0.625\,\boldsymbol{i} + 1.083\,\boldsymbol{j} + 2\,\boldsymbol{k}$$

$$\boldsymbol{a}_D = \boldsymbol{a}_{D'} + \boldsymbol{a}_{D/F} + \boldsymbol{a}_C$$

$$\boldsymbol{a}_{D'} = \dot{\Omega} \times \boldsymbol{r} + \Omega \times (\Omega \times \boldsymbol{r})$$

$$= -200\,\boldsymbol{j} \times (0.1\,\boldsymbol{i} + 0.1732\,\boldsymbol{j}) - 20\,\boldsymbol{j} \times 2\,\boldsymbol{k}$$

$$= 40\,\boldsymbol{i} + 20\,\boldsymbol{k}$$

$$\boldsymbol{a}_C = 2\,\Omega \times \boldsymbol{v}_{D/F} = 2\,(-20\,\boldsymbol{j}) \times (0.625\,\boldsymbol{i} + 1.083\,\boldsymbol{j})$$

$$= 25\,\boldsymbol{k}$$

$$\boldsymbol{a}_D = -32.5\,\boldsymbol{i} + 12.99\,\boldsymbol{j} + 45\,\boldsymbol{k}\ \text{(m/s}^2\text{)}$$

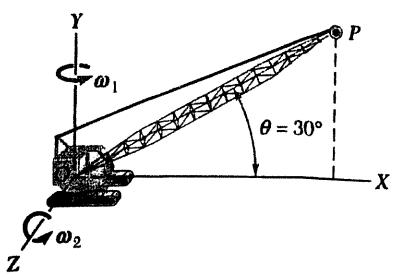
Sample Problem 15.14

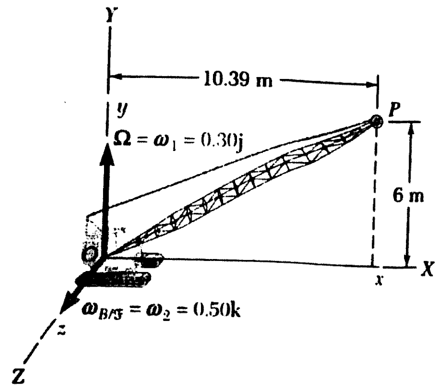
$\omega_1 = 0.3\,\boldsymbol{J} = \Omega$

$\omega_2 = 0.5\,\boldsymbol{k} = \omega_{B/F}$

$OP = 12$ m

determine $\boldsymbol{v}_P$ and $\boldsymbol{a}_P$





choose the cab as the rotating coordinate

then $\quad r = 10.39\,\boldsymbol{i} + 6\,\boldsymbol{j}$

$\Omega = \omega_1 = 0.3\,\boldsymbol{j}$

$\boldsymbol{v}_P = \boldsymbol{v}_{P'} + \boldsymbol{v}_{P/F}$

$= \Omega \times \boldsymbol{r} + \omega_{B/F} \times \boldsymbol{r}$

$= 0.3\boldsymbol{j} \times (10.39\,\boldsymbol{i} + 6\boldsymbol{j}) + 0.5\,\boldsymbol{k} \times (10.39\,\boldsymbol{i} + 6\boldsymbol{j})$

$= -3\,\boldsymbol{i} + 5.2\boldsymbol{j} - 3.12\,\boldsymbol{k}$ (m/s)

$\boldsymbol{a}_P = \boldsymbol{a}_{P'} + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C$

$\boldsymbol{a}_{P'} = \dot{\Omega} \times \boldsymbol{r} + \Omega \times (\Omega \times \boldsymbol{r})$

$= 0 + 0.3\boldsymbol{j} \times 3.12\,\boldsymbol{k} = -0.94\,\boldsymbol{i}$

$\boldsymbol{a}_{P/F} = \omega_{B/F} \times (\omega_{B/F} \times \boldsymbol{r})$

$= 0.5\,\boldsymbol{k} \times (-3\,\boldsymbol{i} + 5.2\boldsymbol{j}) = -2.6\,\boldsymbol{i} - 1.5\,\boldsymbol{j}$

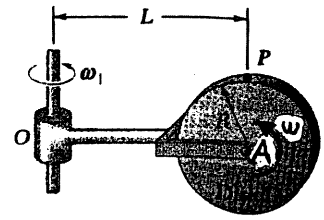
$\boldsymbol{a}_C = 2\,\Omega \times \boldsymbol{v}_{P/F}$

$= 2\,(0.3\boldsymbol{j}) \times (-3\,\boldsymbol{i} + 5.2\boldsymbol{j}) = 1.8\,\boldsymbol{k}$

$\boldsymbol{a}_P = -3.54\,\boldsymbol{i} - 1.5\,\boldsymbol{j} + 1.8\,\boldsymbol{k}$ (m/s²)

the results are the same as in sample problem 15.11

Sample Problem 15.15

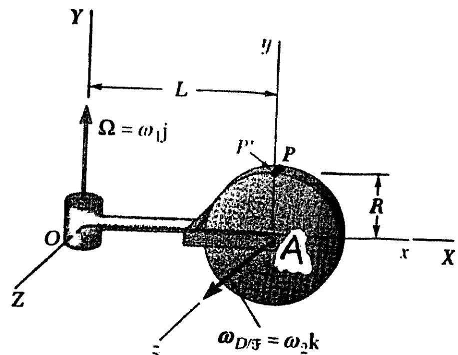


ω_1 and ω_2 are given as constants

determine $\boldsymbol{v}_P$, $\boldsymbol{a}_P$, and ω and α of the disk

choose *Axyz* as the rotating coordinate

then $\Omega = \omega_1\boldsymbol{j}$ and $\omega_{D/F} = \omega_2\,\boldsymbol{k}$



$\boldsymbol{v}_P = \boldsymbol{v}_A + \boldsymbol{v}_{P/A} = \boldsymbol{v}_A + (\boldsymbol{v}_{P'/A} + \boldsymbol{v}_{P/F})_{Axyz}$

$\boldsymbol{v}_A = \omega_1 \times OA = \omega_1\boldsymbol{j} \times L\,\boldsymbol{i} = -\omega_1 L\,\boldsymbol{k}$

$\boldsymbol{v}_{P'/A} = \Omega \times AP' = \omega_1\boldsymbol{j} \times R\boldsymbol{j} = 0$

$\boldsymbol{v}_{P/F} = \omega_2\,\boldsymbol{k} \times R\boldsymbol{j} = -\omega_2 R\,\boldsymbol{i}$

$\boldsymbol{v}_P = -\omega_2 R\,\boldsymbol{i} - \omega_1 L\,\boldsymbol{k}$

$$\boldsymbol{a}_P = [\boldsymbol{a}_{P'}]_{OXYZ} + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C = (\boldsymbol{a}_A + \boldsymbol{a}_{P'/F}) + \boldsymbol{a}_{P/F} + \boldsymbol{a}_C$$

$$\boldsymbol{a}_A + \boldsymbol{a}_{P'/F} = \Omega \text{ x } (\Omega \text{ x } L\,\boldsymbol{i}) + 0$$

$$= \omega_1 \boldsymbol{j} \text{ x } (-\,\omega_1 L\,\boldsymbol{k}) = -\,\omega_1^2 L\,\boldsymbol{i} = \boldsymbol{a}_A$$

$$\boldsymbol{a}_{P/F} = \omega_2 \text{ x } (\omega_2 \text{ x } \boldsymbol{AP})$$

$$= \omega_2 \boldsymbol{k} \text{ x } (-\,\omega_2 R\,\boldsymbol{i}) = -\,\omega_2^2 R\,\boldsymbol{j}$$

$$\boldsymbol{a}_C = 2\,\Omega \text{ x } \boldsymbol{v}_{P/F}$$

$$= 2\,\omega_1 \boldsymbol{j} \text{ x } (-\,\omega_2 R\,\boldsymbol{i}) = 2\,\omega_1\,\omega_2\,\boldsymbol{k}$$

$$\boldsymbol{a}_P = -\,\omega_1^2 L\,\boldsymbol{i} - \omega_2^2 R\,\boldsymbol{j} + 2\,\omega_1\,\omega_2\,\boldsymbol{k}$$

$$\omega_D = \Omega + \omega_{D/F} = \omega_1 \boldsymbol{j} + \omega_2 \boldsymbol{k}$$

$$\alpha_D = (\dot{\omega}_D)_{OXYZ} = (\dot{\omega}_D)_{Axyz} + \Omega \text{ x } \omega_D$$

$$= 0 + \omega_1 \boldsymbol{j} \text{ x } (\omega_1 \boldsymbol{j} + \omega_2 \boldsymbol{k})$$

$$= \omega_1\,\omega_2\,\boldsymbol{i}$$