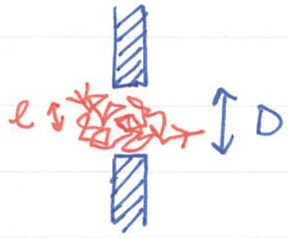
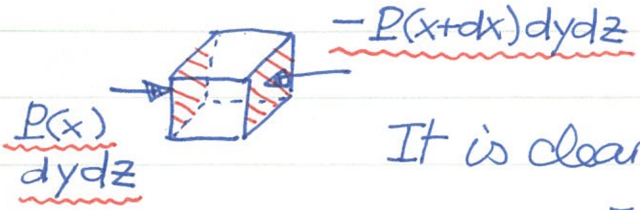
HH0051 Navier-Stokes Equation



$l \ll D$

In the diffusive regime, it is better to describe the transport properties by hydrodynamic approach. We will start with nonviscous hydrodynamics first.

① Euler equation: Consider the dynamics of an infinitesimal cube.



The force due to the pressure is $\vec{F_p} = (F_{px}, F_{py}, F_{pz})$

It is clear from the figure that — minus sign!

$$F_{px} = -[P(x+dx) - P(x)]\,dydz = -\frac{\partial P}{\partial x}\,dxdydz$$

Similar expression for F_{py}, F_{pz}. $\longrightarrow$ $\boxed{\vec{F_p} = -\vec{\nabla}P \cdot dxdydz}$

The mass of the tiny cube is $M = \rho\, dxdydz$ and the external force per unit volume is $\vec{f}_{ex}$

$$\longrightarrow \quad \rho\frac{dv}{dt} = -\vec{\nabla}P + \vec{f}_{ex}$$

Use the same trick to express $\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})$

It leads to the Euler equation:

$$\boxed{\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\vec{v} + \vec{\nabla}P = \vec{f}_{ex}}$$

If the system is close to equilibrium, the Euler equation can be simplified: local velocity $\vec{v}$, all spatial and time derivatives are small! For instance, the divergence of $\rho\vec{v}$

$$\vec{\nabla}\cdot(\rho\vec{v}) = \rho\cdot\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}\rho$$

second order term, can be ignored here ☺

$$\approx \rho\,\vec{\nabla}\cdot\vec{v}.$$

Therefore, we can drop products of gradient terms and just keep the lowest order contributions. The linearized Euler equation now reads:

$\vec{v}, \vec{\nabla}, \frac{\partial}{\partial t}$ first-order term.

$$\boxed{\rho\frac{\partial\vec{v}}{\partial t} + \vec{\nabla}P = \vec{f}_{ex}}$$

drop $\rho\vec{v}\cdot\vec{\nabla}\vec{v}$ term.

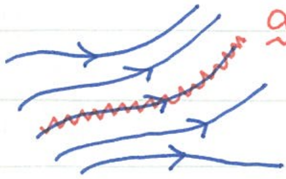
Because of particle conservation, $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$ continuity equation.

Its linearized version is

$$\boxed{\frac{\partial \rho}{\partial t} + \rho\,\vec{\nabla}\cdot\vec{v} = 0}$$

← $\vec{v}\cdot\vec{\nabla}\rho$ dropped (2^{nd} order)

Furthermore, if the energy transfer due to collisions average to zero, the fluid element undergoes adiabatic transformations along a streamline.



adiabatic

$$\frac{d}{dt}(P\rho^{-\gamma}) = 0 \longrightarrow \boxed{\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)(P\rho^{-\gamma}) = 0}$$

Again, dropping $\vec{v}\cdot\vec{\nabla}$ gives the linearized version.

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial t} + \rho\,\vec{\nabla}\cdot\vec{v} = 0 \\ &\frac{\partial}{\partial t}(P\rho^{-\gamma}) = 0 \\ &\rho\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}P = \vec{f}_{ex} \end{aligned}}$$

→ Linearized equations for nonviscous hydrodynamics ☺

⊘ Sound propagation: Taking time derivative of the linearized continuity equation and keeping only the first-order terms, we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \rho\,\vec{\nabla}\cdot\frac{\partial \vec{v}}{\partial t} = 0$$

← drop $\frac{\partial \rho}{\partial t}\cdot\vec{\nabla}\cdot\vec{v}$, product of derivatives → 2^{nd} order term

In the absence of external force $\vec{f}_{ex} = 0$, $\partial\vec{v}/\partial t = -(1/\rho)\,\vec{\nabla}P$.

$$\frac{\partial^2 \rho}{\partial t^2} - \rho\,\vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{\nabla}P\right) = 0 \longrightarrow \boxed{\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P = 0}$$

← just first order

Evaluate $\nabla^2 P$: $\nabla^2 P = \vec{\nabla}\cdot\vec{\nabla}P = \vec{\nabla}\cdot\left[\left(\frac{\partial P}{\partial \rho}\right)_\sigma \vec{\nabla}\rho\right] \approx \left(\frac{\partial P}{\partial \rho}\right)_\sigma \nabla^2\rho$

note that $\left(\frac{\partial P}{\partial \rho}\right)_\sigma = \frac{1}{\rho k_s}$, where k_s is the adiabatic compressibility.

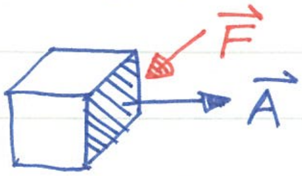
Finally, the density propagation (i.e. sound waves) is described by the following linear wave equation:

$$\nabla^2 P = \frac{1}{\rho k_s}\nabla^2\rho \longrightarrow \boxed{\nabla^2\rho - \frac{1}{c^2}\frac{\partial^2 \rho}{\partial t^2} = 0}$$

speed of sound $c = 1/\sqrt{\rho k_s}$

For ideal gas $\left(\frac{\partial P}{\partial \rho}\right)_\sigma = \gamma\frac{P}{\rho} = \gamma\cdot\frac{n\tau}{nM} = \frac{\gamma\tau}{M} \longrightarrow c = \sqrt{\frac{\gamma\tau}{M}}$

⊘ Navier-Stokes equation: When viscosity is taken into account, the force produced from pressure is no longer normal to the surface. We need a rank two tensor P_{ij}

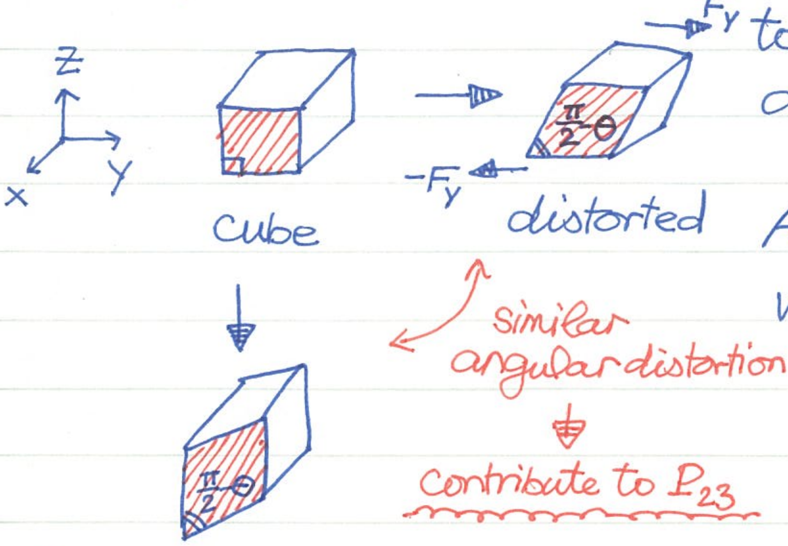


$$\boxed{F_i = -P_{ij}A_j}$$

Einstein summation here!

One can also show that $P_{ij} = P_{ji}$ is symmetric.

Let's try to estimate the component P_{23} of the stress tensor, related to the distortion of the shaded area.



$$\boxed{\frac{d\theta}{dt} = \frac{\frac{\partial u_y}{\partial z}dz\,dt}{dz\cdot dt} = \frac{\partial u_y}{\partial z}}$$

According to $F_y/A_z = -\eta\frac{\partial u_y}{\partial z}$, we notice that the stress tensor

$$\boxed{P_{23} = -\eta\frac{d\theta}{dt}}$$

From the figure, we know that $\frac{\partial u_z}{\partial y}$ causes the same angular distortion. Adding two parts together →

$$\boxed{P_{23} = -\eta\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)}$$

i.e. $P_{ij} = -\eta\left(\partial_i u_j + \partial_j u_i\right)$ for $i \neq j$

The diagonal parts of P_{ij} are harder to derive from drawing pictures → Let's use tensor analysis.

$$P_{ij} = a\,\delta_{ij} + b\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + c\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

a, b, c are some constants.

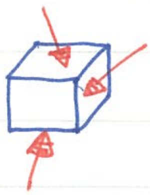
Comparing with our previous derivations: $a = P$ and $b = -\eta$

We need to determine the constant c. Stokes assumed the bulk viscosity is zero, i.e. $\boxed{P = \frac{1}{3}(P_{xx}+P_{yy}+P_{zz})}$ → $P_{ii} = 3P$.

$$P_{ii} = \cancel{P\delta_{ii}} - 2\eta\frac{\partial u_i}{\partial x_i} + c\,\delta_{ii}\frac{\partial u_k}{\partial x_k} = \cancel{3P}$$ (note that $\delta_{ii} = 3$)

→ $\boxed{c = \frac{2}{3}\eta}$ → $$P_{ij} = \delta_{ij}P + \frac{2}{3}\eta\,\delta_{ij}\frac{\partial u_k}{\partial x_k} - \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The force from the stress tensor can be computed,



$$F_x = -\left(\frac{\partial P_{xx}}{\partial x}dx\right)dy\,dz - \left(\frac{\partial P_{xy}}{\partial y}dy\right)dx\,dz$$
$$-\left(\frac{\partial P_{xz}}{\partial z}dz\right)dx\,dy = -\frac{\partial P_{xj}}{\partial x_j}dx\,dy\,dz$$

Compare with previous derivations, we just need to replace $\partial P/\partial x_i$ in Euler equation by $\partial P_{ij}/\partial x_j$:

$$\longrightarrow \boxed{\rho\left(\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right)v_i + \frac{\partial P_{ij}}{\partial x_j} = f_i^{ex}}$$

Navier-Stokes equation.

Or, more explicitly, $\displaystyle \frac{\partial P_{ij}}{\partial x_j} = \frac{\partial P}{\partial x_i} + \frac{2}{3}\eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right) - \eta\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \eta\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)$

$$\longrightarrow \frac{\partial P_{ij}}{\partial x_j} = \vec{\nabla}P - \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta\nabla^2\vec{v}$$

The Navier-Stokes equation now takes the explicit form,

$$\boxed{\rho\left(\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \vec{\nabla}P - \frac{\eta}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \eta\nabla^2\vec{v} = \vec{f}_{ex}}$$

We can study how sound waves propagate in Navier-Stokes equation. The derivations are similar to the previous one,

$$\frac{\partial^2\rho}{\partial t^2} + \rho\,\vec{\nabla}\cdot\left(\frac{\partial\vec{v}}{\partial t}\right) = 0 \longrightarrow \frac{\partial^2\rho}{\partial t^2} - \nabla^2 P + \frac{4}{3}\eta\,\nabla^2(\vec{\nabla}\cdot\vec{v}) = 0$$

$$\longrightarrow \boxed{\frac{\partial^2\rho}{\partial t^2} - c^2\nabla^2\rho - \frac{4\eta}{3\rho_0}\nabla^2\frac{\partial\rho}{\partial t} = 0}$$

additional $\nabla^2\frac{\partial\rho}{\partial t}$ term just like damping ☺

Set $\rho(\vec{r},t) = \rho(t)e^{i\vec{k}\cdot\vec{r}}$. The equation for $\rho(t)$ is simple,

$$\boxed{\frac{\partial^2\rho}{\partial t^2} + \frac{4\eta k^2}{3\rho_0}\frac{\partial\rho}{\partial t} + c^2k^2\rho = 0}$$

compare with damped SHO
$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

The dispersion relation satisfies : $\omega^2 + i\left(\frac{4\eta k^2}{3\rho_0}\right)\omega - c^2k^2 = 0$

The solution for $\omega = \omega(k) = \omega_r + i\omega_I$ now contains imaginary part $\longrightarrow$ Sound wave propagation is damped due to viscosity.

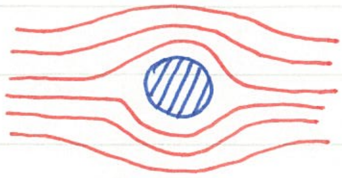
The existence of viscosity introduces a new scale in hydrodynamics.

$$[\eta] = \left[\frac{\text{force/area}}{\text{velocity/length}}\right] = \frac{M}{LT} \longrightarrow \text{Define the Reynolds number}$$

$$\boxed{R \equiv \frac{\rho L u_0}{\eta}}$$

L = characteristic length. ρ = (mass) density

u_0 = flow velocity. η = viscosity coefficient.



$R \ll 1$

Steamline flow

$R \gg 1$

turbulent flow.

The Reynolds number is a good indicator for the appearance of turbulent flows!

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