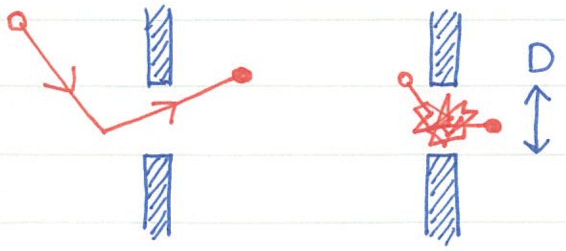
HH0047 Law of Rarefied Gases

Once we realize the existence of mean free path ℓ, the transport is separated into two regimes: effusion and diffusion. To describe molecular effusion, we just need to "count" the particles without collisions. On the other hand, diffusion is naturally described by hydrodynamic approach ☺

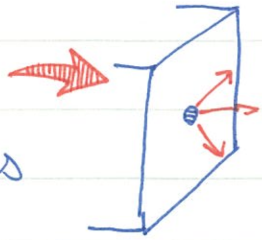


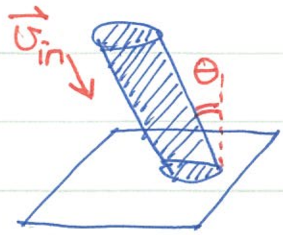
$\lambda \gg D$ effusion.

$\lambda \ll D$ diffusion.

Let's work out several examples to understand effusion better ☺

⓪ Effusion through a hole.



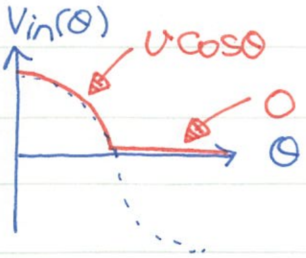


The Maxwell distribution tells us

$$\int_0^\infty dv \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \cdot \frac{1}{4\pi} P_M(v) = 1$$

The flux density of particles can be computed by averaging over the solid angle,

$$J_n = n \int_0^\infty dv \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\ v_{in}\ \frac{1}{4\pi} P_M(v)$$



$$\rightarrow J_n = n \cdot \underbrace{\left[\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^1 d(\cos\theta)\cdot\cos\theta\right]}_{\to \frac{1}{4}} \times \underbrace{\left[\int_0^\infty dv \cdot v P_M(v)\right]}_{\to \bar{c}}$$

Thus, we obtain

$$\boxed{J_n = \frac{1}{4} n \bar{c}}$$

Meanwhile, we can find the velocity dist through the hole:

$$P(v)\,dv = \frac{\cancel{\frac{1}{4}n} \cdot v P_M(v) \cdot dv}{\cancel{\frac{1}{4}n} \int_0^\infty dv\, v P_M(v)} = \frac{v}{\bar{c}} P_M(v)\,dv.$$

Therefore, velocity dist. for effusion through a hole is

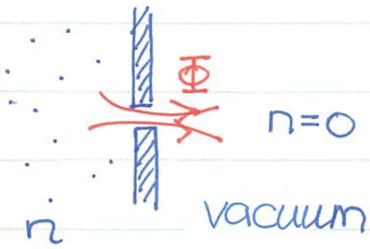
$$\boxed{P(v) = \frac{v}{\bar{c}} P_M(v) \propto v^3 e^{-Mv^2/2\tau}}$$

as taught in previous lecture.

When two types of gas molecules are present, the flux ratio at constant temperature is

$$\boxed{\frac{J_1}{J_2} = \frac{n_1\bar{c}_1}{n_2\bar{c}_2} = \frac{n_1}{n_2}\sqrt{\frac{M_2}{M_1}}}$$ Graham's law

It's quite useful to separate isotopes ☺



Following the "Ohm's law", we can define the conductance of the hole.

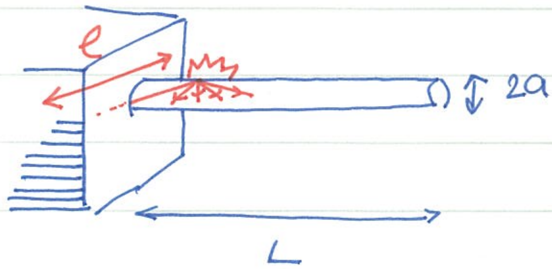
$$\Delta V = IR \longrightarrow I = \frac{\Delta V}{R} = \underline{\underline{G}}\cdot\Delta V$$

conductance.

Similarly, driving the particle flux $\Phi = J_n\cdot A$ is the difference in densities $\Delta n = n - 0 = n$,

$$\longrightarrow \Phi = J_n A = \left(\frac{1}{4}\bar{c}A\right)n \equiv S\,\Delta n\,,\quad \text{i.e.}\ \boxed{S = \frac{1}{4}\bar{c}A}$$

Ⓓ Effusion through a tube. Consider the regime $a \ll \ell \ll L$ and the reflection on the surface inside the tube is diffuse. The striking rate at the surface is $J_n\cdot A$. Thus, the momentum transfer rate is



$$M\langle u\rangle\cdot J_n\cdot A = \Delta P\cdot\pi a^2 \longleftarrow J_n = \frac{1}{4}n\bar{c}\ \ \&\ \ A = 2\pi a L$$

Solve for the drift velocity $\langle u\rangle = \frac{\Delta P}{nM\bar{c}}\,\frac{2a}{L} \propto \Delta P$

The particle flux through the tube,

$$\Phi = n\cdot\langle u\rangle\cdot\pi a^2 = \left(\frac{2\pi a^3\tau}{M\bar{c}L}\right)\frac{\Delta P}{\tau} = \left(\frac{2\pi a^3\tau}{M\bar{c}L}\right)\cdot\Delta n$$

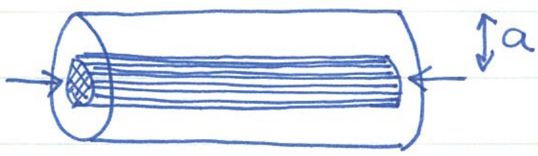
According to the definition of conductance $\Phi = S\cdot\Delta n$

$$\boxed{S_{tube} = \frac{2\pi a^3\tau}{M\bar{c}L}}$$

Note that $\bar{c}^2 = 8\tau/\pi M$, the conductance can be rewritten in more familiar form,

$$S_{tube} = \underbrace{\left(\frac{1}{4}\bar{c}\cdot\pi a^2\right)}_{S_{hole}}\cdot\frac{\pi a}{L} \longrightarrow \boxed{\frac{S_{tube}}{S_{hole}} = \frac{\pi a}{L} \ll 1}$$

② **Diffusion through a tube:** Now consider transport in hydrodynamic regime, the momentum transfer is due to viscosity. Consider a cylindrical part of radius $r < a$.



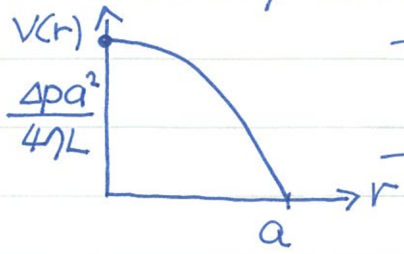
Note that the velocity $V(r)$ vanishes on the surface $r = a$.

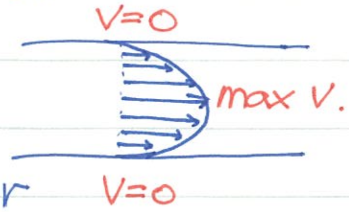
The rate of momentum transfer is balanced:

$$\Delta p \cdot \pi r^2 = \left(-\eta \cdot \frac{dv}{dr}\right) \cdot 2\pi r L \longrightarrow \frac{dv}{dr} = -\frac{\Delta p}{2\eta L} \cdot r$$

It is straightforward to find the velocity $V(r)$.

$$\boxed{V(r) = \frac{\Delta p}{4\eta L}\,(a^2 - r^2)}$$





Integrate over r to find the particle flux:

$$\Phi = n \int_0^a v(r) \cdot 2\pi r\,dr = \frac{n \cdot \Delta p}{4\eta L} \cdot 2\pi \underbrace{\int_0^a (a^2 - r^2)\, r\,dr}_{\frac{1}{4}a^4}$$

$$= \frac{n\pi a^4}{8\eta L} \cdot \Delta p$$

Express the viscosity coefficient $\eta = \frac{1}{3} n m \bar{c} \ell$ to simplify the result.

$$\longrightarrow \Phi = \left(\frac{3\pi a^4 \tau}{8 M \bar{c} \ell L}\right) \cdot \frac{\Delta p}{\tau}$$

The conductance is $\boxed{S_D = \frac{3\pi a^4 \tau}{8 M \bar{c} \ell L}}$

Compare the conductance of the tube in hydrodynamic limit with $S_{hole} = \frac{1}{4}\bar{c} \cdot A$. Making use of $\bar{c}^2 = 8\tau / \pi M$,

$$\boxed{S_D = S_{hole} \cdot \left(\frac{\pi a}{L}\right) \cdot \left(\frac{3a}{16\ell}\right)}$$

small — large.

Thus, S_D and S_{hole} can be of the same order.

But, we can also compare S_D and S_{tube}

$$\boxed{\frac{S_D}{S_{tube}} = \frac{3a}{16\ell} \gg 1} \longrightarrow S_D \gg S_{tube}$$

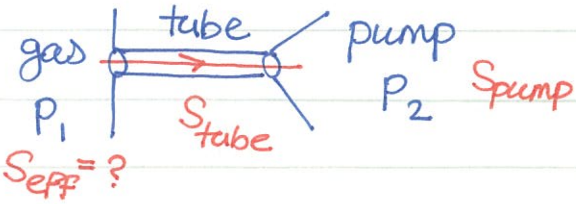
The conductance of the same geometry is much larger in hydrodynamic regime!

④ Speed of a pump: It is defined as the volume pumped per unit time at the intake pressure of the pump.

$$\boxed{S_{pump} \equiv \frac{dV}{dt} = \frac{1}{n}\left(n\frac{dV}{dt}\right) = \frac{1}{n}\Phi}$$

similar to conductance defined before.

Consider the pumping setup:



For given conductance S_{tube} and pumping speed S_{pump}, we would like to know how fast the gas is pumped out...

The flux is constant in the steady state,

$$\boxed{\Phi = n_2 S_{pump} = (n_1 - n_2) S_{tube} = n_1 S_{eff}}$$

Express S_{eff} in terms of S_{pump}, S_{tube}.

$$\frac{S_{eff}}{S_{pump}} = \frac{n_2}{n_1} \quad \text{and} \quad \frac{S_{eff}}{S_{tube}} = \frac{n_1 - n_2}{n_1}$$

Adding these two identities together,

$$S_{eff}\left(\frac{1}{S_{pump}} + \frac{1}{S_{tube}}\right) = 1 \longrightarrow \boxed{\frac{1}{S_{eff}} = \frac{1}{S_{pump}} + \frac{1}{S_{tube}}}$$

This is similar to the addition of resistance in series $R_{eff} = R_1 + R_2$. It is clear that $S_{eff} < S_{pump}$. Therefore, to make the pump efficient, a larger conductance S_{tube} is required ☺

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