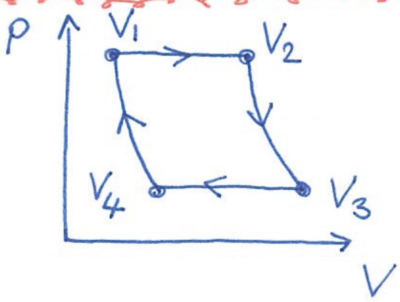
HH0036 Midterm 1 (Thermal Physics II)

1. Photon Carnot engine:



Given $U = aV\tau^4$ and $(\partial\sigma/\partial U)_V = 1/\tau$

$\Rightarrow d\sigma = \frac{1}{\tau}dU = 4aV\tau^2 d\tau$

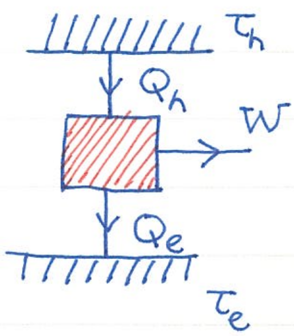
Entropy of a photon gas is

$$\sigma = \frac{4}{3}aV\tau^3 \leftarrow \text{choose } \sigma = 0 \text{ at } \tau = 0.$$

$Q_h = \tau_h(\sigma_2 - \sigma_1)$

$Q_e = \tau_e(\sigma_3 - \sigma_4)$ $\Rightarrow$

$$\boxed{\begin{aligned} Q_h &= \frac{4}{3}a\tau_h^4(V_2 - V_1) \\ Q_e &= \frac{4}{3}a\tau_e^4(V_3 - V_4) \end{aligned}}$$



V_2, V_3 are related by adiabatic process:

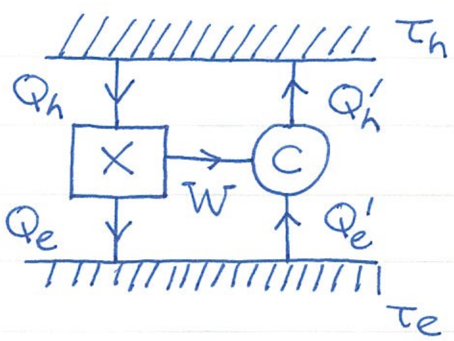
$V_2\tau_h^3 = V_3\tau_e^3$, similarly $V_1\tau_h^3 = V_4\tau_e^3 \rightarrow \tau_h^3(V_2 - V_1) = \tau_e^3(V_3 - V_4)$

The efficiency of the photon Carnot engine is

$$\eta = \frac{W}{Q_h} = 1 - \frac{Q_e}{Q_h} = 1 - \frac{\tau_h^4(V_2 - V_1)}{\tau_e^4(V_3 - V_4)} = \boxed{1 - \frac{\tau_h}{\tau_e} = \eta_c}$$

2. Carnot efficiency:

Reverse the reversible Carnot engine and draw the heat flow diagram.



Energy conservation gives

$$Q_h - Q_h' = Q_e - Q_e'$$

The second law requires $Q_h - Q_h' \geq 0$. According to the definition,

$\eta = \frac{W}{Q_h}$, $\eta_c = \frac{W}{Q_h'}$ $\rightarrow \frac{\eta}{\eta_c} = \frac{Q_h'}{Q_h} \leq 1$ Thus, $\boxed{\eta \leq \eta_c}$

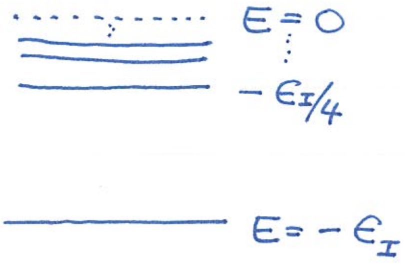
3. Thermal ionization of deuterium: $e^- + D^+ \rightleftharpoons D$

According to Kittel (8.34), (8.35)

$$\prod_j n_j^{\nu_j} = \prod_j n_{Q_j}^{\nu_j} e^{-\nu_j F_j(\mathrm{int})/\tau} \longrightarrow \boxed{\frac{n_{e^-} n_{D^+}}{n_D} \approx n_Q e^{F_D(\mathrm{int})/\tau}}$$

In above, we drop spin degrees of freedom. Meanwhile, $n_Q^{D^+} \approx n_Q^D$.

If we ignore the excitation spectrum of D.



$$F_D(\mathrm{int}) = U_D(\mathrm{int}) - \tau\sigma_D(\mathrm{int}) \approx U_D(\mathrm{int}) = -\epsilon_I$$

Charge neutrality gives $n_{e^-} = n_{D^+}$

$$\longrightarrow \boxed{n_{D^+} \approx \sqrt{n_D n_Q}\; e^{-\epsilon_I/2\tau}}$$

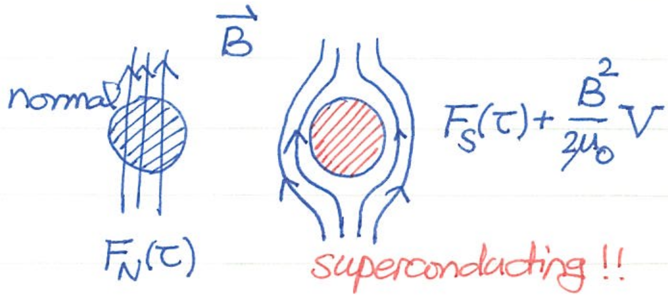
Bonus
$$Z_{int} = e^{\epsilon_I/\tau} + 4e^{\epsilon_I/4\tau} + 9e^{\epsilon_I/9\tau} + \cdots$$
$$= e^{\epsilon_I/\tau}\left(1 + 4e^{-3\epsilon_I/4\tau} + 9e^{-8\epsilon_I/9\tau} + \cdots\right)$$

correction from excitations.

At usual temperatures, $\epsilon_I/\tau \ll 1$, $Z_{int} \approx e^{\epsilon_I/\tau}$. Thus, the free energy is $F_{int} = -\tau \log Z_{int} = -\epsilon_I$.

4. Superconducting transition:

For type-I superconduct,



$$F_N(\tau) - F_S(\tau) = \frac{1}{2\mu_0} V B_c^2(\tau)$$

Taking derivative $-\frac{d}{d\tau}$ on both sides,

$$\sigma_N - \sigma_S = -\frac{1}{\mu_0} V B_c \frac{dB_c}{d\tau} > 0 \longrightarrow \boxed{L = \frac{1}{\mu_0} V \tau B_c \left|\frac{dB_c}{d\tau}\right|}$$

5. Landau theory of phase transition: The Landau free energy is given

$$F_L(\xi, \tau) = \frac{a}{2}(\tau - \tau_c)\xi^2 + \frac{b}{4}\xi^4$$

We assume a, b are positive constants.

Find the order parameter $\frac{\partial F_L}{\partial \xi} = 0 \longrightarrow a(\tau-\tau_c)\xi + b\xi^3 = 0$

$$\boxed{\xi_0 = \begin{cases} 0, & \tau \geq \tau_c \\ \sqrt{\frac{a(\tau_c-\tau)}{b}}, & \tau < \tau_c \end{cases} \qquad F = \begin{cases} 0 & \tau \geq \tau_c \\ -\frac{a^2}{4b}(\tau-\tau_c)^2, & \tau < \tau_c \end{cases}}$$

Substitute into Landau free energy F_L to get the free energy $F(\tau)$. It is clear that $F(\tau)$ and $\partial F/\partial\tau$ are continous, but the 2nd derivative $\partial^2 F/\partial\tau^2$ is not. That's why it is named as "second order" (bad name).

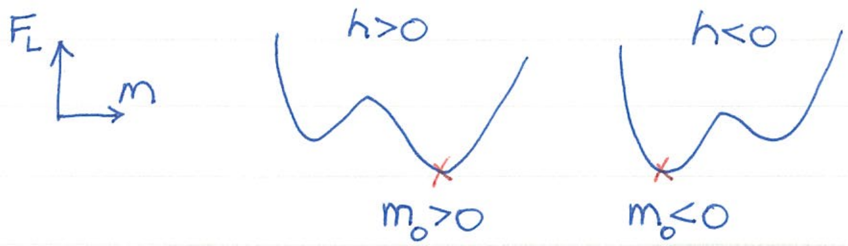
- Discontinuity in $\partial^2 F/\partial\tau^2$ gives rise to a finite jump in specific heat.
- Because $\partial F/\partial\tau$ is continuous, $\sigma(\tau)$ is regular across $\tau = \tau_c \longrightarrow$ no latent heat.
- $\xi_0 = 0$ state ($\tau > \tau_c$) and $\xi_0 \neq 0$ state ($\tau < \tau_c$) do not coexist. $\longrightarrow$ no metastable state
- At $\tau = \tau_c$, ξ^2-term vanishes $\longrightarrow$ infinite correlation length

6. Phase transition in a ferromagnet (Bonus)

$$F_L(m; h, \tau) = -(h-h_c)m + \frac{a}{2}(\tau-\tau_c)m^2 + \frac{b}{4}m^4 \qquad h_c = 0$$

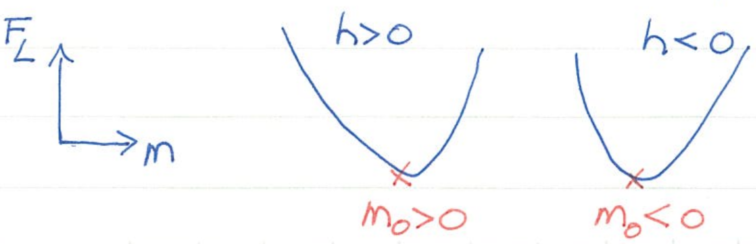
(a) $h=0$, $F_L(m, h=0, \tau)$ is the same as that in Problem 5. Thus, we expect it is a 2nd-order phase transition.

(b) $\tau < \tau_c$, the profile of Landau free energy is shown below



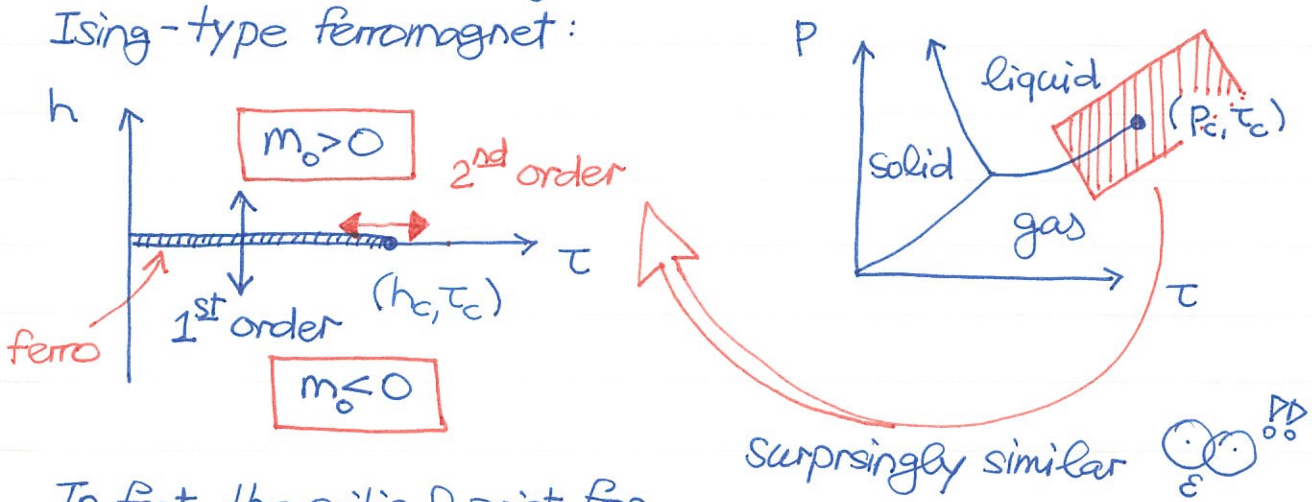
Obviously, it's a first-order phase transition!

(c) Now above the critical temperature $\tau > \tau_c$



Just one minimum moving around $\longrightarrow$ no phase transition!

Collect all results together to cook up the phase diagram for Ising-type ferromagnet:



In fact, the critical point for Ising ferromagnet and that for the liquid-gas system fall into the same universality class and share common critical properties.

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