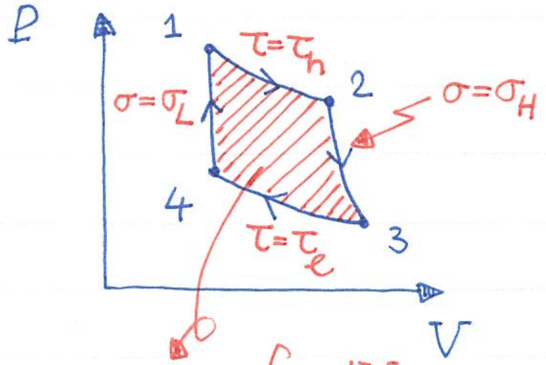
HH0018 Carnot Cycle

Let's consider an ideal heat engine proposed by Carnot. The complete cycle consists of four parts:



area $= \oint p\,dV$
work done by the heat engine

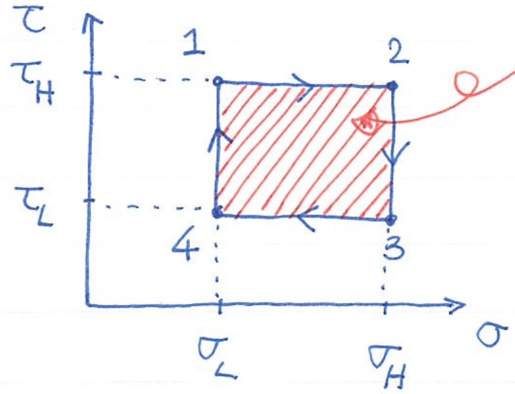
Assume all processes are reversible, the 1st law can be written in the differential form:

$$dU = \tau d\sigma - p\,dV$$

Since the four processes form a closed cycle, $\oint dU = 0$

It's convenient to plot the Carnot cycle in the σ–τ plane:

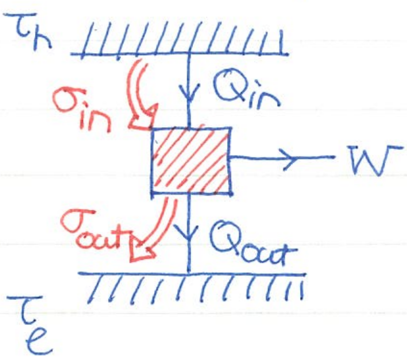
$$\boxed{\oint \tau d\sigma - \oint p\,dV = 0}$$



area $= \oint \tau d\sigma$
$= Q_{in} - Q_{out}$

The above relation implies that area in V-p diagram is the same as that in σ–τ diagram

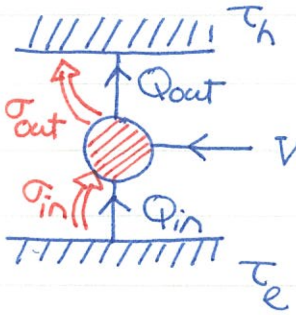
It is quite remarkable that the Carnot heat engine can be represented by the more abstract heat flow diagram:



Because heat transfer occurs at constant temperature, the entropy transfer can be written as

$$\boxed{\sigma_{in} = \frac{Q_{in}}{\tau_h}, \quad \sigma_{out} = \frac{Q_{out}}{\tau_e}}$$

For reversible Carnot cycle, one can run the cycle in opposite direction $\rightarrow$ Carnot refrigerator.



It is of crucial importance to emphasize that

$$\boxed{\sigma_{in} = \sigma_{out}}$$

reversible Carnot cycle

To put our feet on concrete ground, it's helpful to use monoatomic ideal gas as an example.

$1 \rightarrow 2$: $\quad Q_{in} = W_{12} = \int p\,dV = N\tau_h \int \frac{dV}{V} = N\tau_h \log\left(\frac{V_2}{V_1}\right).$

$2 \rightarrow 3$: $\quad W_{23} = U(\tau_h) - U(\tau_\ell) = \frac{3}{2}N(\tau_h - \tau_\ell)$

$3 \rightarrow 4$: $\quad Q_{out} = W_{34} = N\tau_\ell \log\left(\frac{V_3}{V_4}\right)$

$4 \rightarrow 1$: $\quad W_{41} = \frac{3}{2}N(\tau_h - \tau_\ell)$

Making use of the relations, $\tau_\ell V_3^{\frac{2}{3}} = \tau_h V_2^{\frac{2}{3}}$, $\tau_\ell V_4^{\frac{2}{3}} = \tau_h V_1^{\frac{2}{3}}$

$\rightarrow \boxed{\frac{V_3}{V_4} = \frac{V_2}{V_1} \equiv R}$ relates Q_{in} and Q_{out}!

The total work done by the heat engine is ....

$Q_{in} = N\tau_h \log R$ ← heat input

$Q_{out} = N\tau_\ell \log R$ ← waste heat.

work generated → $W = W_{12} + W_{23} - W_{34} - W_{41}$

$= N(\tau_h - \tau_\ell)\log R$

Now we can compute the efficiency of Carnot cycle.

$$\boxed{\eta \equiv \frac{W}{Q_{in}} = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{\tau_\ell}{\tau_h}}$$

It's amazing that the efficiency of a Carnot heat engine only depends on the temperature ratio τ_ℓ/τ_h !!

Carnot showed two famous theorems:

(1) All reversible engines operating between two thermal reservoirs with temperatures τ_ℓ, τ_h have the same efficiency

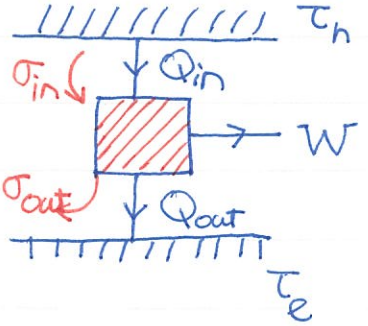
$$\boxed{\eta_c = 1 - \tau_\ell/\tau_h}$$

Carnot efficiency.

(2) Any irreversible engine operating between the same thermal reservoirs has a smaller efficiency,

$$\boxed{\eta \leq \eta_c = 1 - \tau_\ell/\tau_h}$$

Let me use the modern way to prove Carnot's theorems. For reversible Carnot engine, the heat-flow diagram is



$$Q_{in} = Q_{out} + W$$

$$\sigma_{in} = \sigma_{out}$$

only hold for the reversible case

plus

$$\sigma_{in} = \frac{Q_{in}}{\tau_h}$$

$$\sigma_{out} = \frac{Q_{out}}{\tau_\ell}$$

Combine all relations together, it is straightforward to show

$$\eta_c = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{\tau_\ell \sigma_{out}}{\tau_h \sigma_{in}} = 1 - \frac{\tau_\ell}{\tau_h}$$

only depends on temperatures.

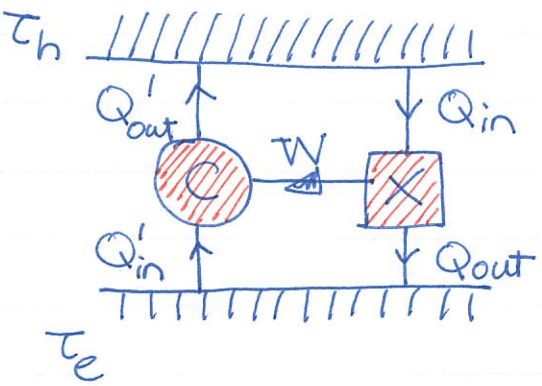
The above calculation can be generalized to the irreversible Carnot engine where $\sigma_{in} \leq \sigma_{out}$

$$\eta = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \left(\frac{\tau_\ell}{\tau_h}\right)\left(\frac{\sigma_{out}}{\sigma_{in}}\right) \leq 1 - \frac{\tau_\ell}{\tau_h} = \eta_c$$

One should notice the beauty of Carnot theorems roots in the simple expression for entropy transfer

It's fun to review the classical proof for Carnot theorems. I will focus on the second theorem here and skip the first one.

Suppose we have a X heat engine with better efficiency $\eta_x > \eta_c$. We can connect the super engine X to a reversed Carnot engine:

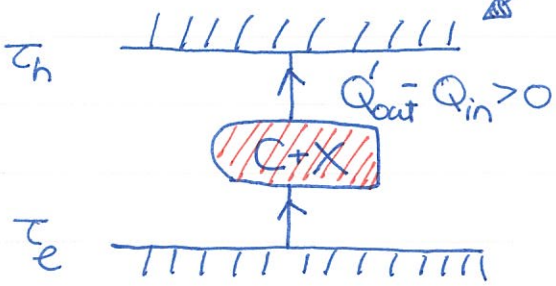


X engine $\rightarrow$ $W = \eta_x Q_{in}$

C engine $\rightarrow$ $W = \eta_c Q'_{out}$

$$\frac{Q'_{out}}{Q_{in}} = \frac{\eta_x}{\eta_c} > 1$$

Making use of the above relation, one can draw the combined heat flow diagram



The C+X engine can pump heat from τ_e to τ_h without external work.... This is against the second law and cannot be true. Thus, $\boxed{\eta_x \leq \eta_c}$

Q: An ideal Otto engine is reversible. Does it have the same efficiency as a reversible Carnot engine?

20120222

清大東院