4.

(a)

$$P(X>20)=\int_{20}^{\infty}\frac{10}{x^2}\,dx=\frac{1}{2}$$

(b)

$$F(y)=\int_{10}^{y}\frac{10}{x^2}\,dx=1-\frac{10}{y} \qquad if\ \ y>10$$

$$F(y)=0 \qquad\qquad if\ \ y\le 10$$

(c)

$$P(X>15)=1-F(15)=1-(1-\frac{10}{15})=\frac{2}{3}\equiv p_1$$

則

Z : 6 個裝置能運作超過 15 小時的個數 $\sim Bin(n=6,p_1)$

$$P(Z\ge 3)=1-P(Z\le 2)=1-\sum_{z=0}^{2}C_z^6(\frac{2}{3})^z(\frac{1}{3})^{6-z}=0.8999$$

$Assumptions$: 每個裝置運作超過15個小時爲獨立進行。

7.

$$\int_0^1 a+bx^2\ dx=1\Rightarrow\left(ax+\frac{b}{3}x^3\right)_{x=0}^{x=1}=1\Rightarrow a+\frac{b}{3}=1$$

$$\int_0^1 x(a+bx^2)\,dx=\frac{3}{5}\Rightarrow\left(\frac{ax^2}{2}+\frac{b}{4}x^4\right)_{x=0}^{x=1}=\frac{3}{5}\Rightarrow\frac{a}{2}+\frac{b}{4}=\frac{3}{5}$$

$$\Rightarrow a=\frac{3}{5}\ ,\ b=\frac{6}{5}$$

10.

(a)

X : 旅客在 7 點與 8 點間到站的時間 $\sim Unif(0,60)$

$P(Goes\ to\ destination\ A)$

$=P(5<X<15\ \ or\ \ 20<X<30\ \ or\ \ 35<X<45\ \ or\ \ 50<X<60)$

$$=\int_5^{15}\frac{1}{60}dx+\int_{20}^{30}\frac{1}{60}dx+\int_{35}^{45}\frac{1}{60}dx+\int_{50}^{60}\frac{1}{60}dx=\frac{2}{3}$$

(b)

Y : 旅客在 7 點10分與 8 點10分間到站的時間 $\sim Unif(10,70)$

$P(Goes\ to\ destination\ A)$

$=P(10<Y<15\ \ or\ \ 20<Y<30\ \ or\ \ 35<Y<45\ \ or\ \ 50<Y<60\ \ or\ 65<Y<70)$

$$=\int_{10}^{15}\frac{1}{60}dy+\int_{20}^{30}\frac{1}{60}dy+\int_{35}^{45}\frac{1}{60}dy+\int_{50}^{60}\frac{1}{60}dy+\int_{65}^{70}\frac{1}{60}dy=\frac{2}{3}$$

11.

X : 在長度爲 L 的線段上，任選一點到某一端點的距離 $\sim Unif(0, L)$

$Case1 : X > L - X$

$$P(\frac{L-X}{X} < \frac{1}{4}) = P(X > \frac{4}{5}L) = \int_{\frac{4}{5}L}^{L} \frac{1}{L}\, dx = \frac{1}{5}$$

$Case2 : X < L - X$

$$P(\frac{X}{L-X} < \frac{1}{4}) = P(X < \frac{1}{5}L) = \int_{0}^{\frac{1}{5}L} \frac{1}{L}\, dx = \frac{1}{5}$$

$\Rightarrow$

$$P(The\ raito\ of\ the\ shorter\ to\ the\ longer\ segment\ is\ less\ than\ \frac{1}{4}) = \frac{2}{5}$$

40.

$X \sim Unif(0,1)$

$f(x) = 1 \qquad 0 < x < 1$

$Y = e^X \ \Rightarrow\ X = \ln Y$

$$|J| = \frac{1}{y}$$

$$h(y) = f(\ln y) \cdot |J| = \frac{1}{y} \qquad 1 < y < e$$

41.

$$\theta \sim Unif(-\frac{\pi}{2}, \frac{\pi}{2})$$

$$f(\theta) = \frac{1}{\pi} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$R = A\sin\theta \ \Rightarrow\ \theta = \sin^{-1}(\frac{R}{A})$$

$$|J| = \frac{1}{A \cdot \sqrt{1 - (\frac{R}{A})^2}}$$

$$h(r) = f(\sin^{-1}(\frac{r}{A})) \cdot |J| = \frac{1}{\pi \cdot A \cdot \sqrt{1 - (\frac{R}{A})^2}} \qquad -A < r < A$$

Theoretical Exercise

2.

$$E(Y)=\int_{-\infty}^{\infty} y\cdot f(y)dy=\int_{0}^{\infty} y\cdot f(y)dy+\int_{-\infty}^{0} y\cdot f(y)dy$$

$$=\int_{0}^{\infty}\int_{0}^{y} dx\, f(y)dy-\int_{-\infty}^{0}\int_{0}^{-y} dx\, f(y)dy$$

$$=\int_{0}^{\infty}\int_{x}^{\infty} f(y)dydx-\int_{0}^{\infty}\int_{-\infty}^{-x} f(y)dydx$$

$$=\int_{0}^{\infty} P(Y>x)dx-\int_{0}^{\infty} P(Y<-x)dx$$

$$=\int_{0}^{\infty} P(Y>y)dy-\int_{0}^{\infty} P(Y<-y)dy$$

3.

$$E[g(X)]=\int_{0}^{\infty} P(g(X)>y)dy-\int_{0}^{\infty} P(g(X)<-y)dy$$

其中

假設 g^{-1} 存在

$$\int_{0}^{\infty} P\{g(X)>y\}dy=\int_{0}^{\infty} P\{X>g^{-1}(y)\}dy=\int_{0}^{\infty}\int_{g^{-1}(y)}^{\infty} f(x)dxdy$$

$$=\int_{0}^{\infty}\int_{0}^{g(x)} f(x)dydx=\int_{0}^{\infty} g(x)f(x)dx$$

$$\int_{0}^{\infty} P\{g(X)<-y\}dy=\int_{0}^{\infty} P\{X<g^{-1}(-y)\}dy=\int_{0}^{\infty}\int_{-\infty}^{g^{-1}(-y)} f(x)dxdy$$

$$=\int_{-\infty}^{0}\int_{0}^{-g(x)} f(x)dydx=\int_{-\infty}^{0} -g(x)f(x)dx$$

Hence

$$E[g(X)]=\int_{0}^{\infty} g(x)f(x)dx-\int_{-\infty}^{0} -g(x)f(x)dx$$

$$=\int_{0}^{\infty} g(x)f(x)dx+\int_{-\infty}^{0} g(x)f(x)dx$$

$$=\int_{-\infty}^{\infty} g(x)f(x)dx$$