Chapter 9 Distributed Forces : Moments of Inertia

9.1 Introduction

in chapter 5, the first moment is defined to determine the centroid of an area of a body

$$Q_x = \int y\, dA$$
$$Q_y = \int x\, dA$$

because some force $\boldsymbol{R}$ depends upon the 2nd moment of inertia $I_x = \int y^2\, dA$ and the polar moment of inertia $J_0 = \int r^2\, dA$ etc., in this chapter, we want to learn how to determine the 2nd moment of inertia of an area

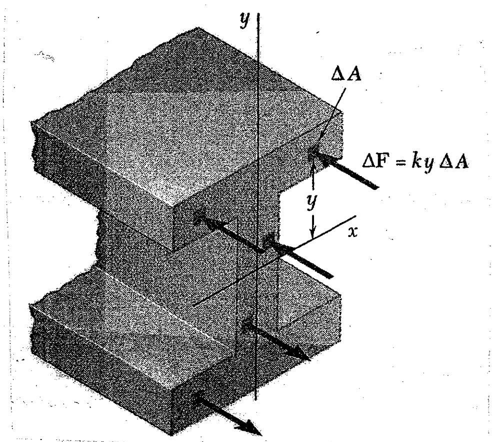
similarly, the mass moments of inertia are also determined

$$I_x = \int y^2\, dA \qquad I_y = \int x^2\, dA \qquad \text{etc.}$$

Moments of Inertia of Areas

9.2 Second Moment, or Moment of Inertia, of an Area

consider a distributed forces $\Delta \boldsymbol{F}$ whose magnitudes depend only upon the element of area ΔA, but also upon the distance from ΔA to some axis



e.g. in pure bending

$$\Delta \boldsymbol{F} = k\, y\, \Delta A$$

where k is a constant, y is the distance from ΔA to an axis passing through the centroid of the section, this axis is known as neutral axis of the section [will be shown in mechanics of materials]

the resultant $\boldsymbol{R}$ can be determined

$$\boldsymbol{R} = \int dF = \int k\, y\, dA = k \int y\, dA$$

$\int y\,dA = \underline{y}\,A = 0$ thus the centroid located on x-axis

for the bending moment M

$$M = \int y\,dF = \int k y^2\,dA = k \int y^2\,dA$$

the term $\int y^2\,dA$ is known as the second moment, or moment of inertia, of the section with respect to the x-axis, and denoted

$$I_x = \int y^2\,dA$$

$I_x > 0$, I_x must be positive

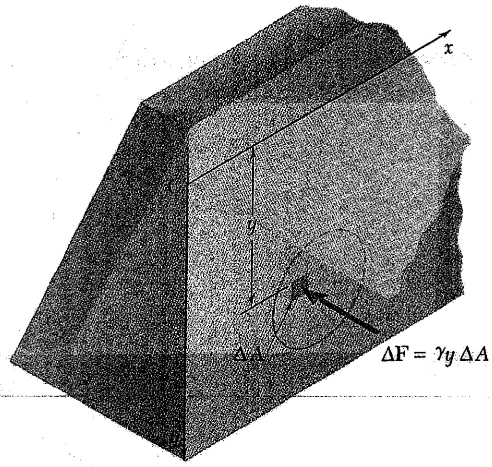
for hydrostatic forces

$$dF = p\,dA$$

where $p = \gamma y$, and the resultant $\boldsymbol{R}$ is

$$\boldsymbol{R} = \int dF = \gamma \int y\,dA$$

and the moment about x-axis is

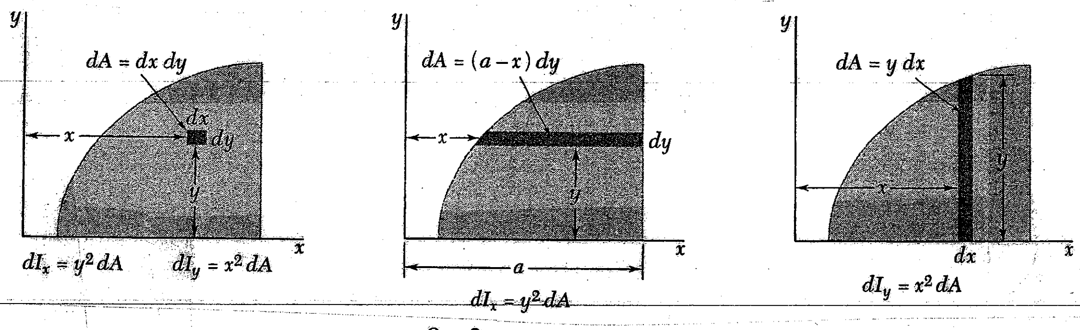


$$dM_x = y\,dF = \gamma y^2\,dA$$
$$M_x = \gamma \int y^2\,dA = \gamma I_x$$

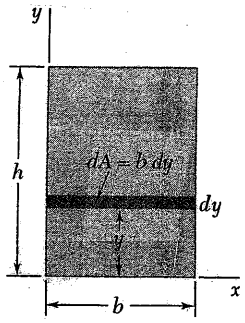
9.3 Determination of the Moment of Inertia of an Area by Integration

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA$$

to computer the moments of inertia, three types of element can be chosen, small element $dx\,dy$, the strip parallel to x-axis and the strip parallel to y-axis



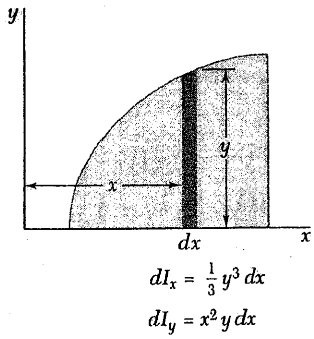
moment of inertia of a rectangular area



$$dI_x = y^2\,dA = b\,y^2\,dy$$

$$I_x = b \int_0^h y^2\,dy = \tfrac{1}{3}\,b\,h^3$$

computing I_x and I_y using the same element strips

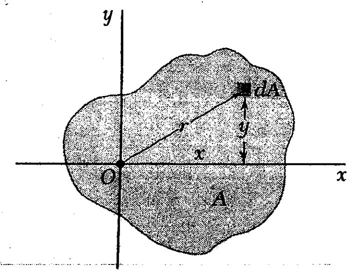


$$dI_x = \tfrac{1}{3}\,y^3\,dx$$

$$dI_y = x^2\,dA = x^2\,y\,dx$$

9.4 Polar Moment of Inertia

the polar moment of inertia of an area with respect to a pole O is defined



$$J_0 = \int r^2\,dA$$

note that $\quad r^2 = x^2 + y^2, \quad$ then

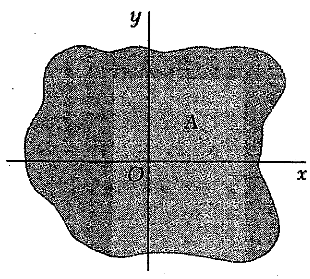
$$J_0 = \int r^2\,dA = \int (x^2 + y^2)\,dA$$

$$= I_x + I_y$$

9.5 Radius of Gyration of an Area

consider an area A which has I_x

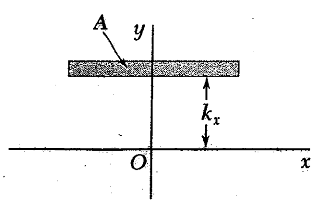
let us imagine that the area concentrate into a strip parallel to x-axis with a distance k_x from the x-axis, if these two areas have same I_x, then



(a)

$$I_x = k_x^2\,A$$

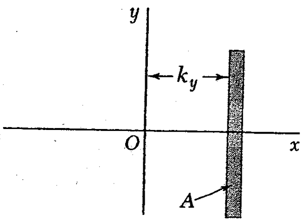
or $\quad k_x = (I_x / A)^{1/2}$

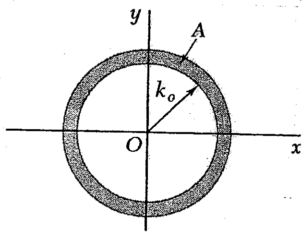


the distance k_x is referred to as the radius of gyration of the area with respect to the x-axis, similarly k_y and k_0 can be defined

$I_y = k_y^2 A \qquad k_y = (I_y / A)^{½}$

$J_0 = k_0^2 A \qquad k_0 = (J_0 / A)^{½}$

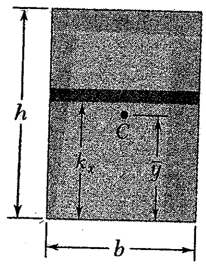




and the relation can be found

$k_0^2 = k_x^2 + k_y^2$

for a rectangular area

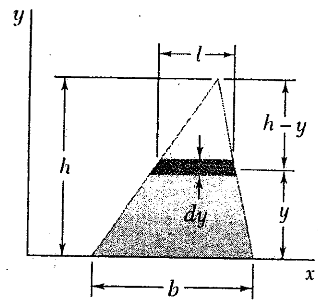


$k_x^2 = I_x / A = (⅓\, b\, h^3) / b\, h = h^2 / 3$

or $\quad k_x = h / \sqrt{3}$

Sample Problem 9.1

determine I_x of the triangle



$dI_x = y^2\, dA = y^2\, l\, dy$

$l / b = (h - y) / h \qquad l = (h - y)\, b / h$

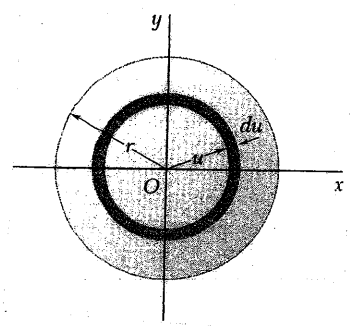
$I_x = \int y^2\, dA = \int_0^h y^2\, [(h - y)\, b / h]\, dy$

$= b/h \int_0^h (y^2\, h - y^3)\, dy = b/h\, [h\, y3 / 3 - y^4 / 4]_0^h$

$= b\, h^3 / 12$

Sample Problem 9.2

Determine J_0 and $I_{diameter}$



$dJ_0 = u^2\, dA = u^2\, 2\pi u\, du$

$J_0 = \int dJ_0 = \int_0^r 2\pi u^3\, du = ½\, \pi\, r^4$

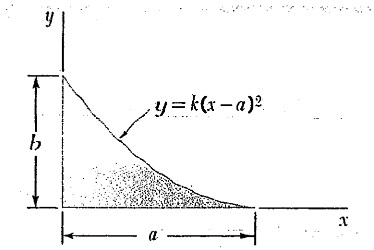
because of symmetry, $\quad I_x = I_y$

$J_0 = I_x + I_y = 2\, I_x = I_{diameter}$

$I_{diameter} = J_0 / 2 = \pi\, r^4 / 4$

Sample Problem 9.3

determine I_x, I_y, k_x, and k_y

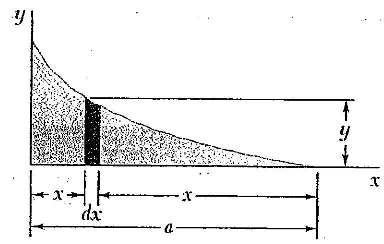


$$y = k(x - a)^2$$

$$y = b \text{ at } x = 0, \; k = b / a^2$$

$$y = b (x - a)^2 / a^2 \qquad A = \tfrac{1}{3} a b$$

moment of inertia I_x



$$dI_x = \tfrac{1}{3} y^3 \, dx = \tfrac{1}{3} [b^3 (x - a)^6 / a^6) \, dx$$

$$I_x = \int dI_x = b^3 / 3a^6 \int_0^a (x - a)^6 \, dx = a \, b^3 / 21$$

moment of inertia I_y

$$dI_y = x^2 \, dA = x^2 \, y \, dx = x^2 \, b \, (x - a)^2 \, dx / a^2$$

$$= b \, (x^4 - {}^2 \, a \, x^3 + a^2 \, x^2) \, dx / a^2$$

$$I_y = \int dI_y = b / a^2 \int_0^a (x^4 - 2 \, a \, x^3 + a^2 \, x^2) \, dx$$

$$= (b / a^2) \, [x^5/5 - a \, x^4/2 + a^2 \, x^3/3]_0^a = a^3 \, b / 30$$

radii of gyration k_x and k_y

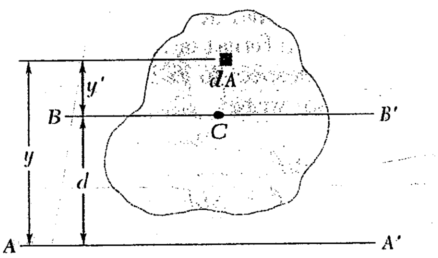
$$k_x = (I_x / A)^{1/2} = [(ab^3/21) / (ab/3)]^{1/2} = b / \sqrt{7}$$

$$k_y = (I_y / A)^{1/2} = [(a^3b/30) / (ab/3)]^{1/2} = a / \sqrt{10}$$

9.6 Parallel-Axis Theorem

consider I of the area about AA' is

$$I = \int y^2 \, dA$$



draw an axis BB', for which $BB' \; // \; AA'$

BB' through the centroid of the area

let $\qquad y = y' + d$

$$I_{AA'} = \int y^2 \, dA$$

$$= \int (y' + d)^2\, dA = \int (y'^2 + 2\,d\,y' + d^2)\, dA$$

$$= \int y'^2\, dA + 2\,d \int y'\, dA + d^2 \int dA$$

$$I_{AA'} = I + A\,d^2$$

I : the moment of inertia of the area with respect to the centroidal axis BB', note that I is the minimum I of the area with respect to a set of parallel axes

$I = k^2 A$ and define $I = k^2 A$

then $k^2 A = k^2 A + A\,d^2$

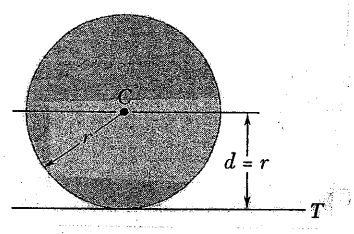
thus $k^2 = k^2 + d^2$

similarly for the polar moment of inertia

$J_0 = J_C + A\,d^2$ $\quad\quad k_0^2 = k_0^2 + d^2$

where J_C is the polar moment of inertia about its centroid C, and d is the distance between O and C

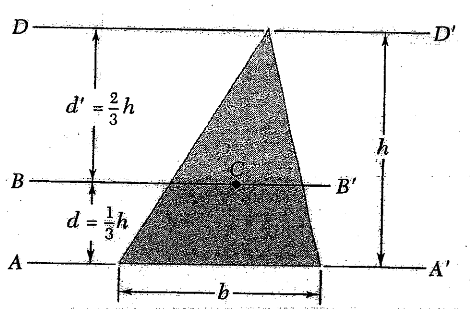
example 1 for a circular area



$$I_T = I + A\,d^2$$

$$= \pi r^4 / 4 + \pi r^2 r^2 = 5\,\pi r^4 / 4$$

example 2 for a triangular area



$$I_{AA'} = I_{BB'} + A\,d^2$$

$$I_{BB'} = I_{AA'} - A\,d^2$$

$$= bh^3 / 12 - \tfrac{1}{2}\,bh\,(h/3)^2 = bh^3 / 36$$

$$I_{DD'} = I_{BB'} + A\,d'^2$$

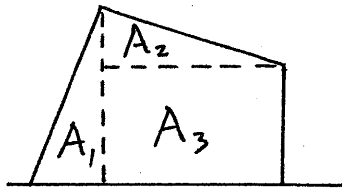
$$= bh^3 /36 + \tfrac{1}{2}\,bh\,(2h/3)^2 = bh^3 / 4$$

note that $I_{DD'} \neq I_{AA'} + A\,h^2$

because AA' is not the axis pass through the centroid C

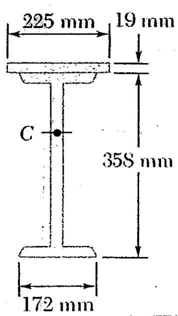
9.7 Moment of Inertia of Composite Areas

consider a composite area A made of several component areas $A_1, A_2, \ldots$



then $\quad (I_x)_A = (I_x)_{A_1} + (I_x)_{A_2} + (I_x)_{A_3}$

Sample Problem 9.4



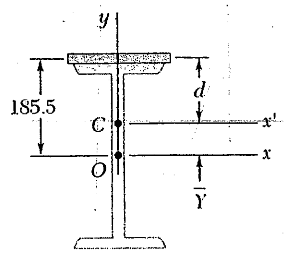
determine I, k about axis through C and // the plate

first, we need to determine the location of C

for the plate $A = 225 \text{ x } 19 = 4275 \text{ mm}^2$

$\bar{y} = ½\, 358 + ½\, 19 = 188.5 \text{ mm}$

	A (mm²)	$\bar{y}$ (mm)	$\bar{y}A$ (mm³)
plate	4275	188.5	805,837.5
wide-flange	7230	0	0
Σ	11505		805,837.5



$\bar{Y} = \Sigma \bar{y} A / \Sigma A = 70.04 \text{ mm}$

for the wide-flange section

$I_{x'} = I_x + A\bar{Y}^2 = 160 \text{ x } 10^6 + 7230 \text{ x } 70.04^2$

$= 195.47 \text{ x } 10^6 \text{ mm}^4$

for the plate

$I_{x'} = \bar{I}_x + A d^2 = 225 \text{ x } 19^3 / 12 + 4275 \text{ x } (188.5 - 70.04)^2$

$= 60.12 \text{ x } 10^6 \text{ mm}^4$

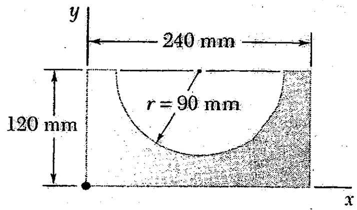
for the composite area

$I_{x'} = 195.47 \text{ x } 10^6 + 60.12 \text{ x } 10^6 = 255.6 \text{ x } 10^6 \text{ mm}^4$

radius of gyration

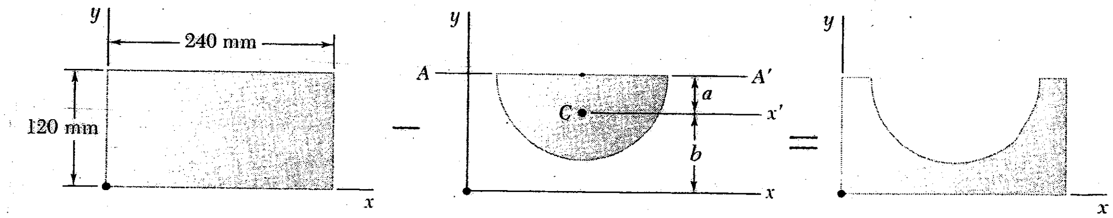
$k_{x'}^2 = I_{x'} / A = 255.6 \text{ x } 10^6 / 11505$

$k_{x'} = 149.1 \text{ mm}$



Sample Problem 8.5

determine I_x of the area



moment of inertia of rectangle

$$I_x = bh^3 / 3 = 240 \text{ x } 120^3 / 3 = 138.2 \text{ x } 10^6 \text{ mm}^4$$

moment of inertia of the half circle

$$a = 4r / 3\pi = 4 \text{ x } 90 / 3\pi = 38.2 \text{ mm}$$

$$b = 120 - a = 81.8 \text{ mm}$$

$$I_{AA'} = \pi r^4 / 8 = \pi\, 90^4 / 8 = 25.76 \text{ x } 10^6 \text{ mm}^4$$

$$A = \pi r^2 / 2 = \pi\, 90^2 / 2 = 12.72 \text{ x } 10^3 \text{ mm}^2$$

$$\underline{I}_{x'} = I_{AA'} - A\, a^2 = 25.76 \text{ x } 10^6 - 12.72 \text{ x } 10^3 \text{ x } 38.2^2$$

$$= 7.20 \text{ x } 10^6 \text{ mm}^4$$

$$I_x = \underline{I}_{x'} + A\, b^2 = 7.20 \text{ x } 10^6 + 12.72 \text{ x } 10^3 \text{ x } 81.8^2$$

$$= 92.3 \text{ x } 10^6 \text{ mm}^4$$

moment of inertia of given area

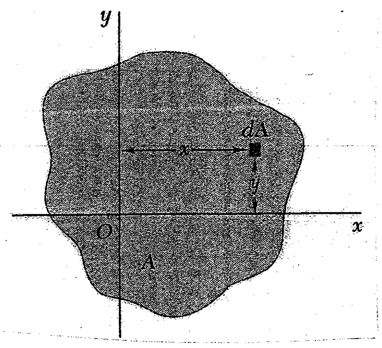
$$I_x = 138.2 \text{ x } 10^6 - 92.3 \text{ x } 10^6 = 45.9 \text{ x } 10^6 \text{ mm}^4$$

9.8 Product of Inertia

the integral

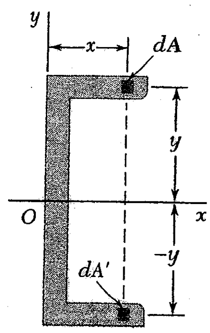
$$I_{xy} = \int x\, y\, dA$$

is known the product of inertia of the area with respect to x and y axes



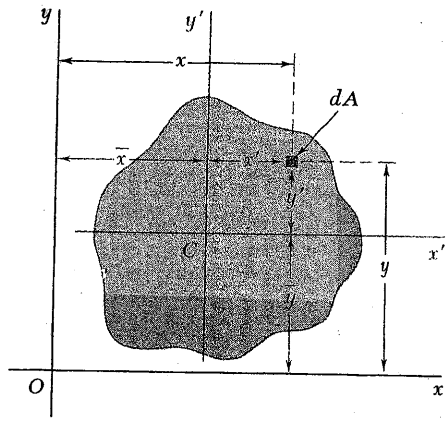
I_{xy} may be either positive or negative

when one or both of the x and y axes are axes of symmetry of the area A, then



$$I_{xy} = 0$$

a parallel-axis theorem similar to the moment of inertia can be derived, we draw centroidal axes x' and y' which are parallel to x and y axes



$$x = x' + \bar{x} \qquad y = y' + \bar{y}$$

$$I_{xy} = \int x\,y\,dA = \int (x' + \bar{x})(y' + \bar{y})\,dA$$

$$= \int x'y'\,dA + \bar{y}\int x'\,dA + \bar{y}\int x'\,dA$$

$$+ \bar{y}\int x'\,dA$$

$$= \bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A$$

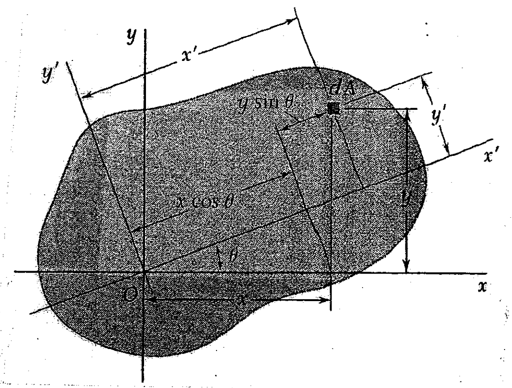
note that the first integrals with respect to centroidal axes are zero

9.9 Principal Axes and Principal Moments of Inertia

consider the area A and x, y axes

$$I_x = \int y^2\,dA \qquad I_y = \int x^2\,dA$$

$$I_{xy} = \int x\,y\,dA$$



denote x' and y' axes are obtained by rotating the original axes by an angle θ

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

$$I_{x'} = \int y'^2\,dA = \int (y\cos\theta - x\sin\theta)^2\,dA$$

$$= \cos^2\theta\int y^2\,dA - 2\sin\theta\cos\theta\int x\,y\,dA + \sin^2\theta\int x^2\,dA$$

$$= I_x\cos^2\theta - 2\,I_{xy}\sin\theta\cos\theta + I_y\sin^2\theta$$

similarly

$$I_{y'} = I_x \sin^2\theta + 2\, I_{xy} \sin\theta \cos\theta + I_y \cos^2\theta$$

$$I_{x'y'} = I_x \sin\theta \cos\theta + I_{xy} (\cos^2\theta - \sin^2\theta) - I_y \sin\theta \cos\theta$$

the trigonometric relations

$$\sin 2\theta = 2 \sin\theta \cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\cos^2\theta = (1 + \cos 2\theta) / 2$$

$$\sin^2\theta = (1 - \cos 2\theta) / 2$$

the equations for moments and product of inertia can be rewritten

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

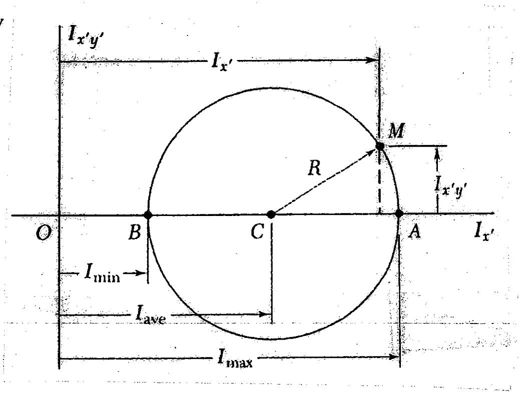
$$I_{x'y'} = \frac{I_x + I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

we observe that

$$I_{x'} + I_{y'} = I_x + I_y = J_0$$

the above equations are the parametric equation of a circle, eliminate θ and it is obtained

$$(I_{x'} - \frac{I_x + I_y}{2})^2 + I_{x'y'}^2 = (\frac{I_x - I_y}{2})^2 + I_{xy}^2$$



setting $\quad I_{ave} = (I_x + I_y) / 2$

$$R = [(\frac{I_x - I_y}{2})^2 + I_{xy}^2]^{1/2}$$

then $\quad (I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$

this is the equation of a circle with radius R and centered at $(I_{ave}, 0)$

the two points A and B has the maximum and minimum values of I with $I_{xy} = 0$

thus, the values θ_m of θ which correspond to the points A and B may be obtained

for $I_{xy} = 0 \Longrightarrow \theta_m$

$$\tan 2\theta_m = -2 I_{xy} / (I_x - I_y)$$

$\theta_m = +$, clockwise $\qquad \theta_m = -$, counterclockwise

$2\,\theta_m$ are 180^o apart, thus θ_m are 90^o apart

one of them corresponds to point A with respect to which the moment of inertia is maximum, the other one corresponds point B with respect to which the moment of inertia is minimum

thus the axes with respect to the maximum and minimum I are perpendicular, these two axes thus defined as the principal axes of the area about O, and I_{max} and I_{min} are called the principal moments of inertia of the area about O

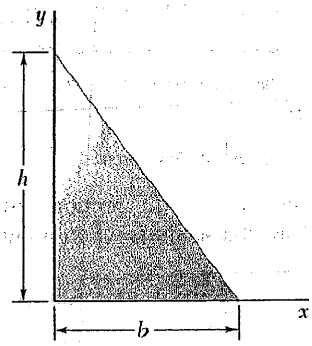
$$I_{max/min} = I_{ave} \pm R$$

$$I_{max/min} = \frac{(I_x + I_y)}{2} \pm [(\frac{I_x - I_y}{2})^2 + I_{xy}^2]^{1/2}$$

if O is chosen to coincide with the centroid of the area, any axes through O is a centroidal axis, the two principal axes are called the principal centroidal axes of the area

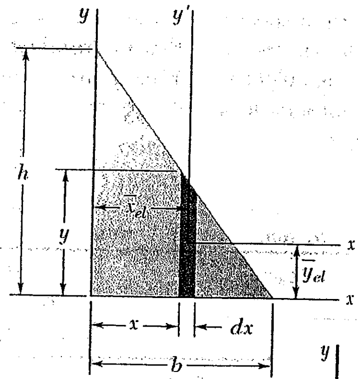
Sample Problem 9.6

determine I_{xy} and $I_{x''y''}$ of the right triangle where x'' and y'' are centroidal axes



$$dI_{xy} = dI_{x'y'} + \underline{x}_{el}\,\underline{y}_{el}\,dA = x\,(y/2)\,y\,dx$$

$$= \tfrac{1}{2}\,x\,y^2\,dx$$



$y = h\,(1 - x/b)$

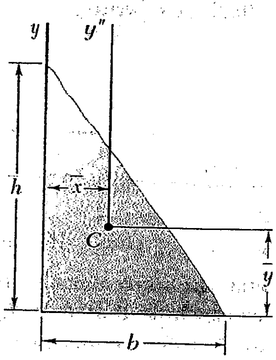
$I_{xy} = \int dI_{xy} = \int_b^a \tfrac{1}{2}\, x\, h^2\,(1 - x/b)^2\, dx$

$= b^2 h^2 / 24$

$I_{xy} = I_{x"y"} + \underline{x}\,\underline{y}\,A$

$I_{x"y"} = b^2 h^2 / 24 - (b/3)\,(h/3)\,bh/2$

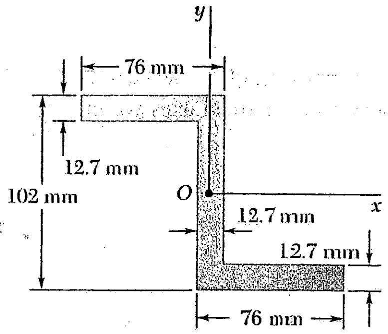
$= - b^2 h^2 / 72$



Sample Problem 9-7

$I_x = 4.32 \times 10^6\ mm^4$ $\quad I_y = 2.9 \times 10^6\ mm^4$

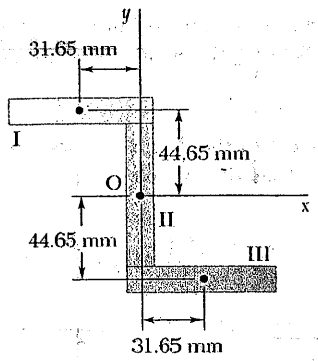
determine the principal axes and principal I



we want to find I_{xy} first

$I_{xy} = I_{x"y"} + \underline{x}\,\underline{y}\,A$

rectangle	A(mm²)	$\underline{x}$(mm)	$\underline{y}$(mm)	$\underline{x}\,\underline{y}\,A$(mm⁴)
I	965	-31.65	44.65	-1.36×10^6
II	973	0	0	0
III	965	-31.65	44.65	-1.36×10^6



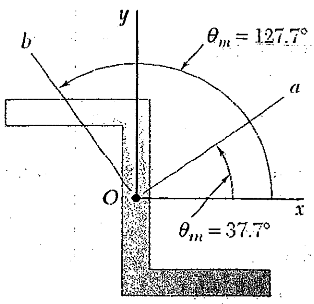
$\Sigma\, \underline{x}\,\underline{y}\,A = - 2.72 \times 10^6\ mm^4$

$I_{xy} = - 2.72 \times 10^6\ mm^4$

$$\tan 2\theta_m = - \frac{2\, I_{xy}}{I_x - I_y} = - \frac{2 \times (- 2.72 \times 10^6)}{(4.32 - 2.9) \times 10^6} = 3.83$$

$\therefore \quad 2\theta_m = 75.4^{o}$ and 255.4^{o}

$\theta_m = 37.7^{o}$ and 127.7^{o}



$$I_{max/min} = \frac{(I_x + I_y)}{2} \pm \left[\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2\right]^{½}$$

$$= \frac{(4.32 + 2.9) \times 10^6}{2} \pm \left[\left(\frac{4.32 - 2.9}{2}\right)^2 + (-2.72)^2\right]^{½} \times 10^6$$

$$I_{max} = 6.42 \times 10^6\ mm^4 = I_a \qquad (\theta_m = 37.7^o)$$

$$I_{min} = 0.8 \times 10^6\ mm^4 = I_b \qquad (\theta_m = 127.7^o)$$

this conclusion can be verified by substituting $\theta_m = 37.7^o$ into the transformation equation to get I_{max}

9.10 Mohr' s Circle for Moments and Products of Inertia

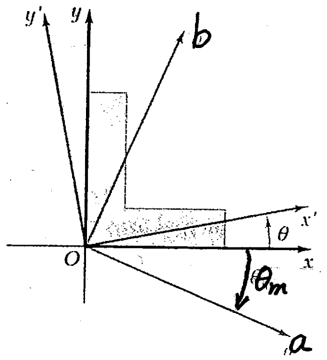
the circle in last section to illustrate the relation between moments and product of inertia is known as the Mohr' s circle

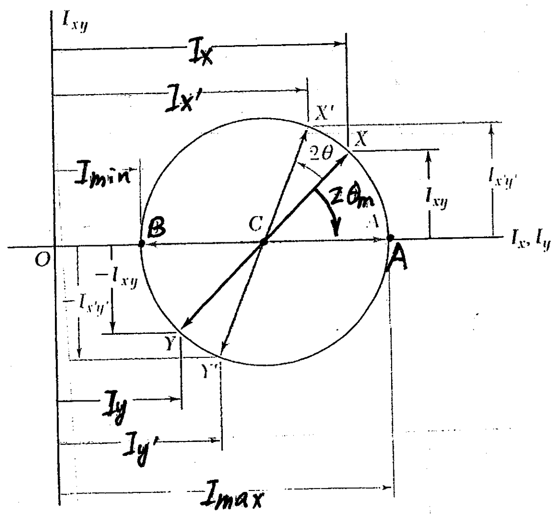
if I_x, I_y, I_{xy} of an area about x and y axes are known, Mohr' s circle may be used to determine graphically

(a) principal axes and principal moments of inertia

(b) the moments and product of inertia of any other pair of perpendicular axes

for a given area, I_x, I_y, I_{xy} are known, the points X and Y can be plotted on the Mohr' s circle, at the point X', the moment of inertia is





$$I_{x'} = R\cos(2\theta + 2\theta_m) + (I_x + I_y)/2$$

$$= (I_x + I_y)/2 + R\cos 2\theta_m(\cos 2\theta - \tan 2\theta_m \sin 2\theta)$$

$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - \frac{I_x - I_y}{2}\,\frac{I_{xy}}{(I_x - I_y)/2}\sin 2\theta$$

$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{x'y'} = R \sin(2\theta + 2\theta_m)$$
$$= R \cos 2\theta_m (\sin 2\theta + \tan 2\theta_m \cos 2\theta)$$
$$= \frac{I_x - I_y}{2} \sin 2\theta + \frac{I_x - I_y}{2} \frac{I_{xy}}{(I_x - I_y)/2} \cos 2\theta$$
$$= \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

the procedure to construct the Mohr' s circle

1. plot the points $X(I_x, I_{xy})$ and $Y(I_x, -I_{xy})$
2. use XY as a diameter to draw a circle, that is Mohr' s circle
3. from X to A, it is rotated clockwise by $2\theta_m$, that means from x-axis to a-axis, it is rotated clockwise by θ_m
4. from X to X', it is rotated counterclockwise by 2θ, that means from x-axis to x' -axis, it is rotated clockwise by θ
5. the coordinate of the points are

$X(I_x, I_{xy})$ $Y(I_x, -I_{xy})$ $C(I_{ave}, 0)$

$X'(I_{x'}, I_{x'y'})$ $Y'(I_{x'}, -I_{x'y'})$

$A(I_a, 0) = I_{max}$ $B(I_b, 0) = I_{min}$

a- and b- axes are principal axes of the area

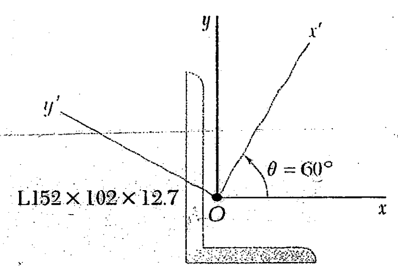
Sample Problem 9-8

$I_x = 7.2 \times 10^6\ mm^4$ $I_y = 2.59 \times 10^6\ mm^4$

$I_{xy} = -2.54 \times 10^6\ mm^4$

determine (a) principal axes, (b) principal I

(c) $I_{x'}$, $I_{y'}$ and $I_{x'y'}$

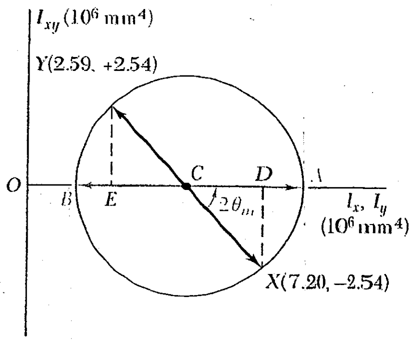


drawing the Mohr' s circle

$$I_{ave} = OC = (I_x + I_y) / 2 = 4.895 \times 10^6\ mm^4$$

$$CD = (I_x - I_y) / 2 = 2.305 \times 10^6\ mm^4$$

$R = (CD^2 + DX^2)^{½} = 3.43 \times 10^6\ mm^4$

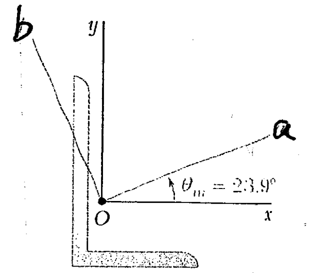


(a) principal axes

$\tan 2\theta_m = DX\,/\,CD = 2.54\,/\,2.305 = 1.102$

$2\theta_m = 47.8°$ $\quad\quad \theta_m = 23.9°$

Oa axis is rotated 23.9° counterclockwise from x-axis



(b) principal moments of inertia

$I_{max} = OA = I_{ave} + R = 8.33 \times 10^6\ mm^4$

$I_{min} = OB = I_{ave} - R = 1.47 \times 10^6\ mm^4$

(c) to determine $I_{x'}$, $I_{y'}$ and $I_{x'y'}$

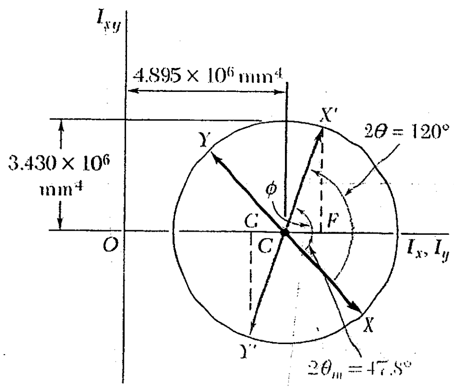
$\angle XCX' = 120°$

$\phi = 120° - 47.6° = 72.2°$

$I_{x'} = I_{ave} + R\cos 72.2° = 5.94 \times 10^6\ mm^4$

$I_{y'} = 2\,I_{ave} - I_{x'} = 3.85 \times 10^6\ mm^4$

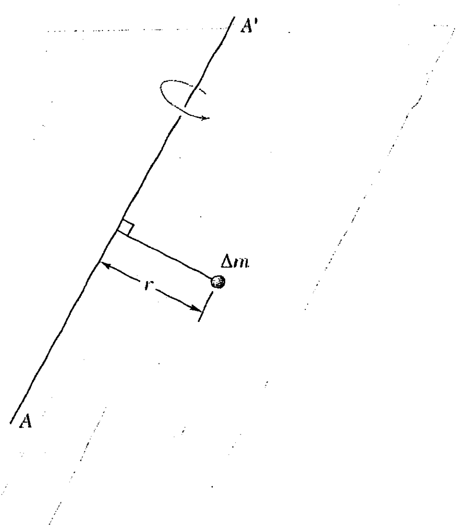
$I_{x'y'} = R \sin 72.2° = 3.27 \times 10^6\ mm^4$



Moments of Inertia of Masses

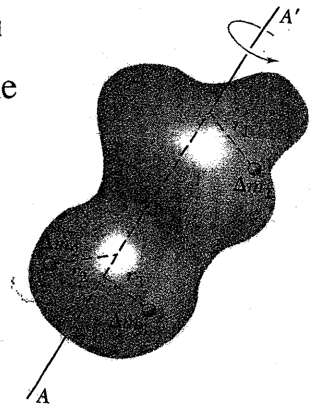
9.11 Moment of Inertia of Mass

consider a small mass Δm with a distance r to an axis AA', the product $r^2\,\Delta m$ is called the moment of inertia of the mass with respect to the axis, i.e.



$$\Delta I = r^2\,\Delta m$$

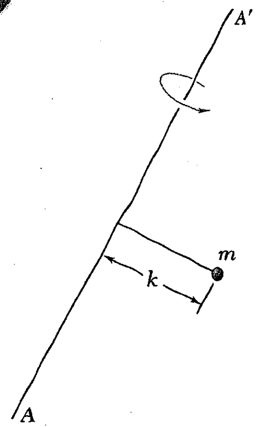
consider now a mass m, the sum of $r_1^2 \, \Delta m_1 + r_1^2 \, \Delta m_1 + \cdots$ is the moment of inertia of the mass with respect to AA'



$$I = \int r^2 \, dm$$

radius of gyration k of a body is defined

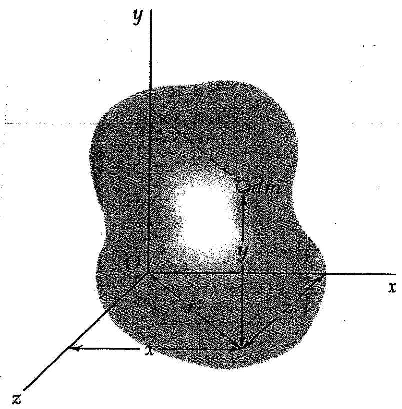
$$I = k^2 m \qquad \text{or} \qquad k = (I / m)^{1/2}$$



k represents the distance at which the entire mass of the body should be concentrated if its moment of inertia of mass with respect to AA' is to remain unchanged

the unit of mass moment of inertia is kg-m^2

the moment of inertia of a body with respect to a coordinate axis can be expressed



$$I_y = \int r^2 \, dm = \int (x^2 + z^2) \, dm$$

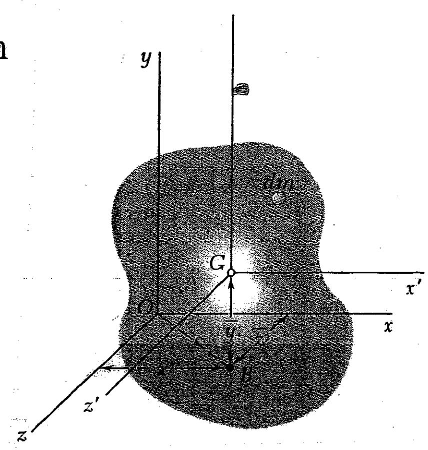
similarly

$$I_x = \int (y^2 + z^2) \, dm$$

$$I_z = \int (x^2 + y^2) \, dm$$

9.12 Parallel-Axis Theorem

consider x', y' and z' axes pass through the center of gravity of the body G, then



$$x = x' + \underline{x}$$

$$y = y' + \underline{y}$$

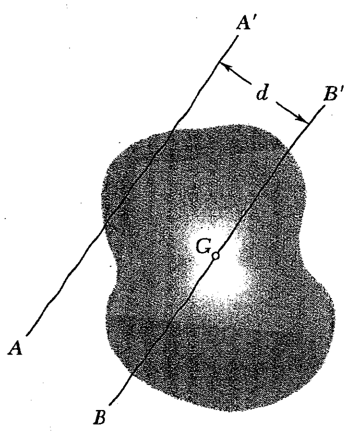
$$z = z' + \underline{z}$$

I_x can be expressed as follows

$$I_x = \int (y^2 + z^2)\, dm = \int [(y' + \underline{y})^2 + (z' + \underline{z})^2]\, dm$$
$$= \int (y'^2 + z'^2)\, dm + 2\underline{y} \int y'\, dm + 2\underline{z} \int z'\, dm + (\underline{y}^2 + \underline{z}^2) \int dm$$
$$= \underline{I}_{x'} + m(\underline{y}^2 + \underline{z}^2)$$

and $I_y = \underline{I}_{y'} + m(\underline{x}^2 + \underline{z}^2)$

$I_z = \underline{I}_{z'} + m(\underline{x}^2 + \underline{y}^2)$



denoting by d the distance between an arbitrary axis AA' and a parallel centroidal axis BB', then

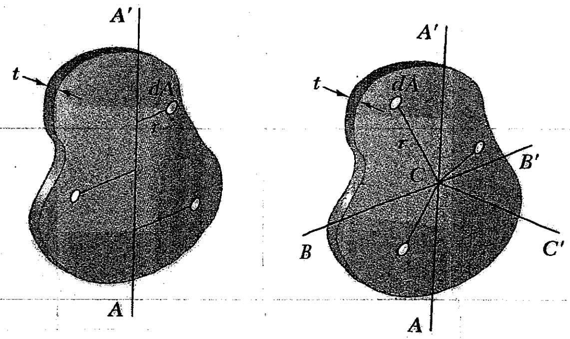
$$I_{AA'} = \underline{I}_{BB'} + m\, d^2$$

and $k_{AA'} = \underline{k}_{BB'} + d^2$

9.13 Moments of Inertia of Thin Plates

consider a thin plate of uniform thickness t and density ρ

$$I_{AA',mass} = \int r^2\, dm$$



since $dm = \rho\, t\, dA$, then

$$I_{AA',mass} = \rho\, t \int r^2\, dA = \rho\, t\, I_{AA',area}$$

similarly

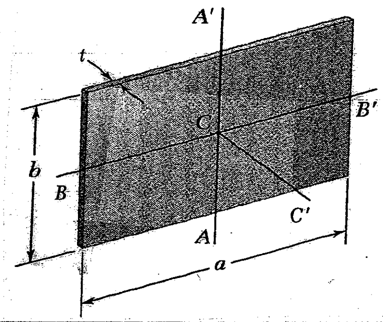
$$I_{BB',mass} = \rho\, t\, I_{BB',area}$$
$$I_{CC',mass} = \rho\, t\, J_{C,area}$$

recalling the relation $J_{C,area} = I_{AA',area} + I_{BB',area}$, then for the thin plate

$$I_{CC',mass} = I_{AA',mass} + I_{BB',mass}$$

for a rectangular plate

$$I_{AA',mass} = \rho\, t\, I_{AA',area} = \rho\, t\, (a^3 b / 12)$$
$$I_{BB',mass} = \rho\, t\, I_{BB',area} = \rho\, t\, (a\, b^3 / 12)$$

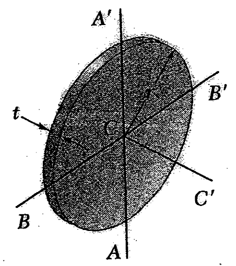


since $m = \rho\, a\, b\, t$, then

$I_{AA',mass} = m\,a^2 / 12 \qquad I_{AA',mass} = m\,b^2 / 12$

and $\qquad I_{CC',mass} = m\,(a^2 + b^2) / 12$

for a circular plate



$I_{AA',mass} = \rho\,t\,I_{AA',area} = \rho\,t\,(\pi\,r^4 / 4)$

$I_{BB',mass} = I_{AA',mass}$

$m = \rho\,t\,\pi\,r^2$

$I_{BB',mass} = I_{AA',mass} = m\,r^2 / 4$

$I_{BB',mass} = m\,r^2 / 2$

9.14 Determination of the Moment of Inertia of a Three-Dimensional Body by Integration

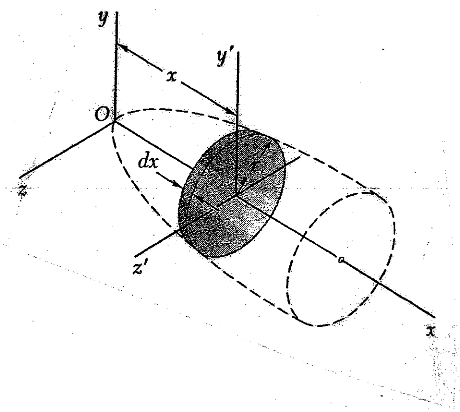
consider the density of the homogeneous body is ρ

$$I = \int r^2\,dm = \rho \int r^2\,dV$$

it will generally be to perform a triple integration

if the body possesses two planes of symmetry, it is usually possible to determine I by a single integral

in the case of bodies of revolution, the element of mass would be a thin disk, use the result of I for a thin plate and the parallel axis theorem, the moment of inertia of the body be expressed



$dm = \rho\pi r^2\,dx$

$dI_x = \frac{1}{2}\,r^2\,dm$

$dI_y = dI_{y'} + x^2\,dm = (\frac{1}{4}\,r^2 + x^2)\,dm$

$dI_z = dI_{z'} + x^2\,dm = (\frac{1}{4}\,r^2 + x^2)\,dm$

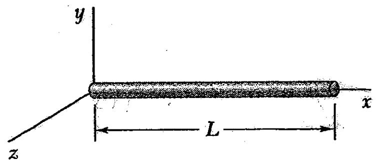
9.15 Moments of Inertia of Composite Bodies

the moment of inertia with respect to a given axis of a body made by several of simple shapes may be obtained by computing the moments of inertia of its component part about the desired axis and adding them together

I for common shapes are listed in figure 9.28 on page 496

Sample Problem 9.9

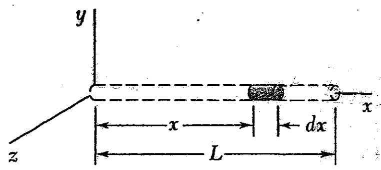
determine I_y of the rod of length L and mass m



$$dm = (m / L)\, dx$$

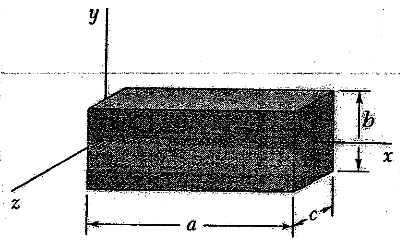
$$I_y = \int r^2\, dm = \frac{m}{L} \int_0^L x^2\, dx$$

$$= m L^2 / 3$$



Sample Problem 9.10

determine I_z of the homogeneous rectangular prism

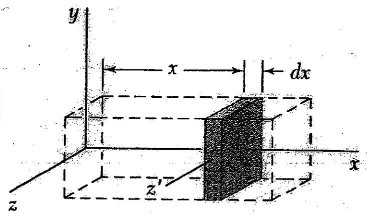


$$dm = \rho\, b\, c\, dx$$

$$dI_{z'} = b^2\, dm / 12$$

$$dI_z = dI_{z'} + x^2\, dm$$

$$= b^2\, dm / 12 + x^2\, dm$$



$$I_z = \int dI_z = \int_0^a (b^2/12 + x^2)\, \rho\, b\, c\, dx$$

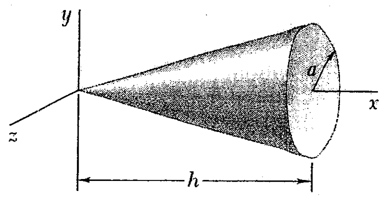
$$= \rho\, a\, b\, c\, (b^2/12 + a^2/3) = m\, (b^2/12 + a^2/3)$$

if $b \ll a$, $I_z = m\, a^2 / 3$, same as sample problem 9.9

Sample Problem 9.11

determine I_x, I_y, and I_{y}'' of the right circular cone

where $y'' \,//\, y$, y'' pass through G

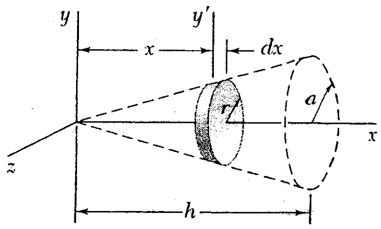


$r = a\,x / h$

$dm = \rho\,\pi\,r^2\,dx = \rho\,\pi\,(a/h)^2\,dx$

a. moment of inertia I_x

$dI_x = \frac{1}{2} r^2\,dm = \frac{1}{2} (ax/h)^2\,\rho\pi(a/h)^2\,dx$



$I_x = \frac{1}{2}\,\rho\pi(a/h)^4 \int_0^h x^4\,dx = \rho\pi a^4 h / 10$

$m = \rho\,V = \frac{1}{3}\,\rho\pi a^2 h$

$I_x = 3\,m\,a^2 / 10$

b. moment of inertia I_y

$dI_{y'} = \frac{1}{4} r^2\,dm$

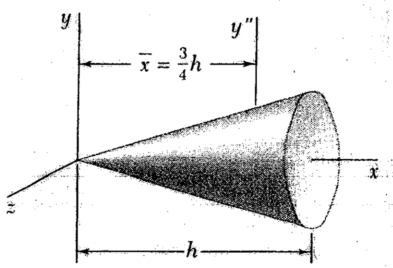
$dI_y = dI_{y'} + x^2\,dm = (\frac{1}{4} r^2 + x^2)\,dm$

$= [\frac{1}{4} (ax/h)^2 + x^2]\,\rho\pi(a/h)^2\,dx$

$$I_y = \rho\pi \frac{a^2}{h^2} (\frac{1}{4} \frac{a^2}{h^2} + 1) \int_0^h x^4\,dx$$

$$= \rho\pi \frac{a^2}{h^2} (\frac{1}{4} \frac{a^2}{h^2} + 1) \frac{h^5}{5} = \frac{3}{5} m (\frac{1}{4} a^2 + h^2)$$

c. moment of inertia $I_{y''}$



$I_y = I_{y''} + m\,\bar{x}^2 \qquad \bar{x} = 3\,h / 4$

$I_{y''} = I_y - m\,\bar{x}^2$

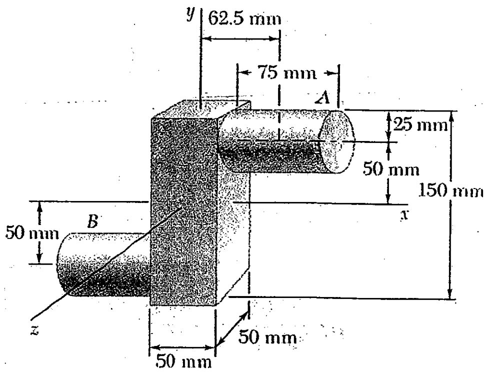
$$= \frac{3}{5} m (\frac{1}{4} a^2 + h^2) - m (\frac{3h}{4})^2$$

$= (3/20)\,m\,(a^2 + \frac{1}{4} h^2)$

Sample Problem 9.12

determine I_x, I_y, and I_z of the body

$\rho = 7850$ kg/m^3



computation of masses

prism

$V = 0.05 \times 0.05 \times 0.15 = 3.75 \times 10^{-4}\ m^3$

$m = \rho V = 7850 \times 3.75 \times 10^{-4} = 2.94$ kg

each cylinder

$V = \pi\, 0.025^2 \times 0.075 = 1.473 \times 10^{-4}\ m^3$

$m = \rho V = 7850 \times 1.473 \times 10^{-4} = 1.156$ kg

moments of inertia

prism

$I_x = I_z = 2.94\,(0.15^2 + 0.05^2) / 12 = 6.125 \times 10^{-3}$ kg-m²

$I_y = 2.94\,(0.05^2 + 0.05^2) / 12 = 1.225 \times 10^{-3}$ kg-m²

each cylinder

$I_x = \frac{1}{2} ma^2 + m\underline{y}^2 = \frac{1}{2}\, 1.156 \times 0.025^2 + 1.156 \times 0.05^2$

$= 3.251 \times 10^{-3}$ kg-m²

$I_y = m\,(3a^2 + L^2) / 12 + m\underline{y}^2$

$= 1.156\,(3 \times 0.025^2 + 0.075^2) / 12 + 1.156 \times 0.0625^2$

$= 5.238 \times 10^{-3}$ kg-m²

$I_z = m\,(3a^2 + L^2) / 12 + m\,(\underline{x}^2 + \underline{y}^2)$

$= 1.156\,(3 \times 0.025^2 + 0.075^2) / 12 + 1.156 \times (0.0625^2 + 0.05^2)$

$= 8.128 \times 10^{-3}$ kg-m²

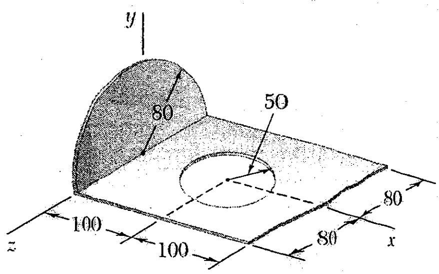
entire body

$I_x = 6.125 \times 10^{-3} + 2 \times 3.251 \times 10^{-3} = 12.63 \times 10^{-3}$ kg-m²

$I_y = 1.225 \times 10^{-3} + 2 \times 5.238 \times 10^{-3} = 11.70 \times 10^{-3}$ kg-m²

$I_z = 6.125 \times 10^{-3} + 2 \times 8.128 \times 10^{-3} = 22.4 \times 10^{-3}$ kg-m²

Sample Problem 9.13

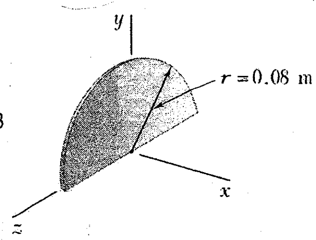


for the thin plate $t = 4$ mm $\rho = 7850$ kg/m^3

determine I_x, I_y, and I_z of the body

computation of masses

semicircular plate

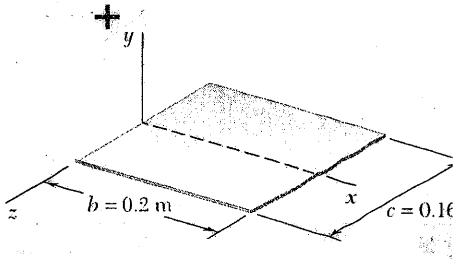


$$V_1 = \tfrac{1}{2}\pi r^2 t = \tfrac{1}{2}\pi\, 0.8^2 \times 0.004 = 40.21 \times 10^{-6}\ \text{m}^3$$

$$m_1 = \rho V_1 = 7850 \times 40.21 \times 10^{-6} = 0.3156\ \text{kg}$$

+

rectangular plate



$$V_2 = 0.2 \times 0.16 \times 0.004 = 128 \times 10^{-6}\ \text{m}^3$$

$$m_2 = \rho V_2 = 7850 \times 128 \times 10^{-6} = 1.005\ \text{kg}$$

circular plate

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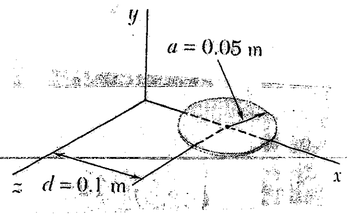

$$V_3 = \pi a^2 t = \pi\, 0.5^2 \times 0.004 = 31.42 \times 10^{-6}\ \text{m}^3$$

$$m_3 = \rho V_3 = 7850 \times 31.42 \times 10^{-6} = 0.2466\ \text{kg}$$

moments of inertia

semicircular plate

$$I_x = \tfrac{1}{2} m_1 r^2 = 1.01 \times 10^{-3}\ \text{kg-m}^2$$

$$I_y = I_z = \tfrac{1}{4} m_1 r^2 = 0.505 \times 10^{-3}\ \text{kg-m}^2$$

rectangular plate

$$I_x = m_2 c^2 / 12 = 2.144 \times 10^{-3}\ \text{kg-m}^2$$

$$I_z = m_2 b^2 / 3 = 13.4 \times 10^{-3}\ \text{kg-m}^2$$

$$I_y = I_x + I_z = 14.544 \times 10^{-3}\ \text{kg-m}^2$$

circular plate

$$I_x = \tfrac{1}{4} m_3 a^2 = 0.154 \times 10^{-3}\ \text{kg-m}^2$$

$$I_y = \tfrac{1}{2} m_3 a^2 + m_3 d^2 = 2.774 \times 10^{-3} \text{ kg-m}^2$$

$$I_z = \tfrac{1}{4} m_3 a^2 + m_3 d^2 = 2.62 \times 10^{-3} \text{ kg-m}^2$$

entire body

$$I_x = (1.01 + 2.144 - 0.154) \times 10^{-3} = 3.00 \times 10^{-3} \text{ kg-m}^2$$

$$I_y = (0.505 + 15.544 - 2.772) \times 10^{-3} = 13.28 \times 10^{-3} \text{ kg-m}^2$$

$$I_z = (0.505 + 13.4 - 2.62) \times 10^{-3} = 11.29 \times 10^{-3} \text{ kg-m}^2$$

9.16 Moment of Inertia of a Body with Respect to an Arbitrary Axis through *O*. Mass Products of Inertia

9.17 Ellipsoid of Inertia. Principal Axes of Inertia

9.18 Determination of the Principal Axes and Principal Moments of Inertia of a Body of Arbitrary Shape