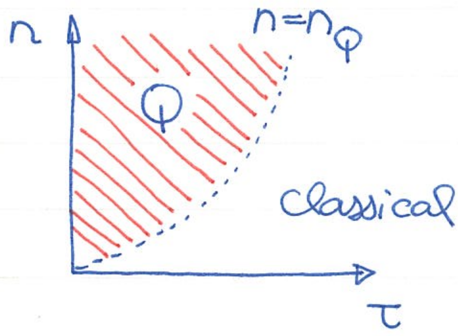
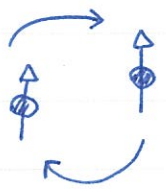
HH0012 Fermi Gas



The quantum and classical regimes are (roughly) separated by $n=n_Q$, where $n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{\frac{3}{2}} \propto \tau^{\frac{3}{2}}$.

$$\boxed{\text{Fermions : half-integral spin } \left(S=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\dots\dots\right)}$$



exchange gives (-1) factor

$$\Psi(\vec{r}_1 s_1 ; \vec{r}_2 s_2) = -\Psi(\vec{r}_2 s_2 ; \vec{r}_1 s_1)$$

But, two fermions tightly bound together

What's the quantum statistics?

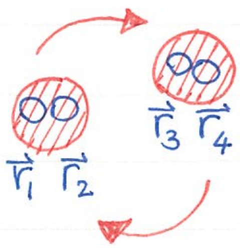
method 1: spin addition. $S_1=\frac{1}{2}$, $S_2=\frac{1}{2}$ $\longrightarrow$ $\boxed{S=0,1}$

Thus, the composite particle is a boson.

method 2:

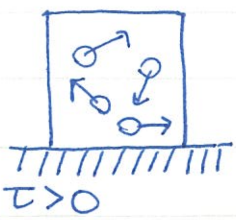
$$\boxed{\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4) = (-1)^2\,\Psi(\vec{r}_3,\vec{r}_4,\vec{r}_1,\vec{r}_2)}$$

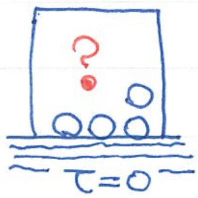
$(+1)$, boson!



rules: F+F $\longrightarrow$ B, F+B $\longrightarrow$ F, B+B $\longrightarrow$ B.

Now we focus on non-interacting fermion system - Fermi Gas ☺
The most striking property of a fermion gas is its large kinetic energy even at $\tau=0$



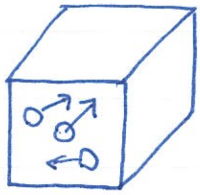


What happens to particles at $\tau=0$?

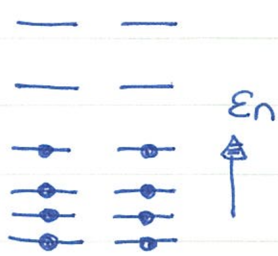
$$\langle \tfrac{1}{2}mv_x^2 \rangle = \langle \tfrac{1}{2}mv_y^2 \rangle = \langle \tfrac{1}{2}mv_z^2 \rangle = \tfrac{1}{2}\tau \to 0$$

All particles stop moving at $\tau=0$?

And the kinetic energy is zero?



single-particle orbitals $\varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$

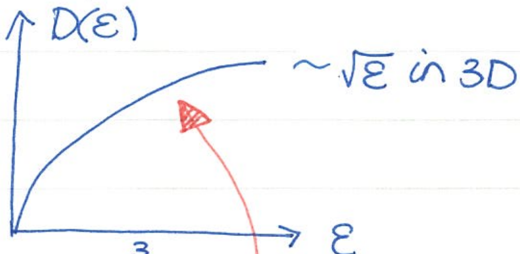


$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{n_F\pi}{L}\right)^2 \text{ is the largest energy of filled orbitals at } \tau = 0 \text{ !!}}$

In previous lecture, we obtain the relation:

$$\boxed{\sum_n (\cdots) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int d\varepsilon \sqrt{\varepsilon}}$$

Because the above relation is very useful, we introduce the notion $\longrightarrow$ density of states $D(\varepsilon)$.



$$\boxed{\underbrace{2}_{\text{spin-}\frac{1}{2}} \times \sum_n (\cdots) = \int d\varepsilon\, D(\varepsilon)}$$

By comparison, $D(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\varepsilon}$

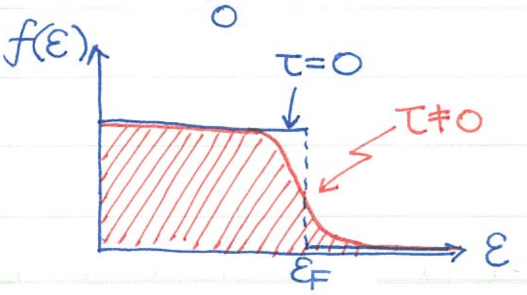
Comment: $D(\varepsilon)\cdot d\varepsilon$ is nothing but the number of orbitals within the energy range $(\varepsilon, \varepsilon + d\varepsilon)$.

$$\boxed{\begin{aligned} &\text{Total number of particles } N = \sum_n f(\varepsilon_n) = \int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon) \\ &\text{Total energy } \quad U = \sum_n \varepsilon_n f(\varepsilon_n) = \int_0^\infty d\varepsilon\, D(\varepsilon)\, \varepsilon f(\varepsilon) \end{aligned}}$$

At $\tau = 0$, $f(\varepsilon_n) = \frac{1}{e^{(\varepsilon_n - \mu)/\tau} + 1} = \begin{cases} 1, & \varepsilon_n < \mu(0) \equiv \varepsilon_F \\ 0 & \varepsilon_n > \mu(0) \equiv \varepsilon_F \end{cases}$

The above expressions simplify.

$$N = \int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon) \quad , \quad U(\tau=0) \equiv U_0 = \int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon)\, \varepsilon > 0$$



If $\tau \ll \varepsilon_F$, the Fermi distribution is still close to the step function, slightly smeared around $\varepsilon \approx \varepsilon_F$.

① heat capacity of electron gas.

We are interested in how energy changes with respect to τ.

$$\boxed{\Delta U = U(\tau) - U(0) = \int_0^\infty d\varepsilon\ \varepsilon\, D(\varepsilon) f(\varepsilon) - \int_0^{\varepsilon_F} d\varepsilon\ \varepsilon\, D(\varepsilon).}$$

Because the particle number is conserved,

$$N = \int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon). \quad \longleftarrow \ N(\tau) = N(0), \text{ of course!}$$

Massage the conservation law into the following identity.

$$\boxed{\int_0^{\varepsilon_F} d\varepsilon\ [1 - f(\varepsilon)]\, D(\varepsilon) - \int_{\varepsilon_F}^\infty d\varepsilon\ f(\varepsilon) D(\varepsilon) = 0}$$

multiplied by ε_F ⟶ add to ΔU expression

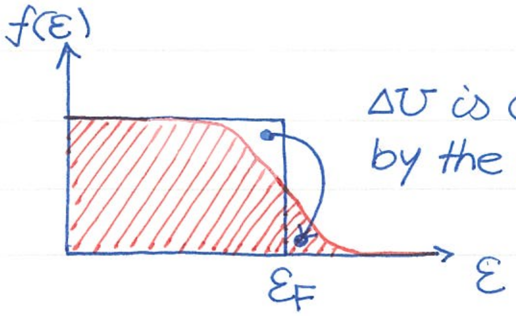
$$\Delta U = \int_0^{\varepsilon_F} d\varepsilon\ (-\varepsilon)[1 - f(\varepsilon)] D(\varepsilon) + \int_{\varepsilon_F}^\infty d\varepsilon\ \varepsilon f(\varepsilon) D(\varepsilon)$$

$$+ \int_0^{\varepsilon_F} d\varepsilon\ \varepsilon_F [1 - f(\varepsilon)] D(\varepsilon) + \int_{\varepsilon_F}^\infty d\varepsilon\ (-\varepsilon_F) f(\varepsilon) D(\varepsilon)$$

$$\boxed{\begin{aligned}\Delta U = &\int_0^{\varepsilon_F} d\varepsilon\ (\varepsilon_F - \varepsilon)[1 - f(\varepsilon)] D(\varepsilon) \\ &+ \int_{\varepsilon_F}^\infty d\varepsilon\ (\varepsilon - \varepsilon_F) f(\varepsilon) D(\varepsilon)\end{aligned}}$$

move electrons below to Fermi level.

move electrons at Fermi level to higher energy.



ΔU is caused by the immigration.

Because both terms are positive, $\Delta U > 0$. This is what we expect, otherwise the ground state is unstable.

note that $$\frac{df}{d\tau} = \frac{-1}{\left[e^{(\varepsilon-\mu)/\tau} + 1\right]^2}\, e^{(\varepsilon-\mu)/\tau} \cdot \left(-\frac{\varepsilon - \mu}{\tau^2}\right)$$

In the limit $\tau \ll \varepsilon_F$, $\mu(\tau) \approx \mu(0) = \varepsilon_F$.

$$\frac{df}{d\tau} \approx \frac{1}{\left[e^{(\varepsilon-\varepsilon_F)/\tau}+1\right]^2}\, e^{(\varepsilon-\varepsilon_F)/\tau} \left(\frac{\varepsilon-\varepsilon_F}{\tau^2}\right)$$ substitute into ΔU ....

The specific heat (coming from non-interacting electrons) is

$$C_{el} = \frac{dU}{d\tau} = \int_0^\infty d\varepsilon\, (\varepsilon-\varepsilon_F) \frac{df}{d\tau} D(\varepsilon)$$ → roughly $D(\varepsilon_F)$

$$= \tau D(\varepsilon_F) \int_{-\varepsilon_F/\tau}^{\infty} dx\, \frac{x^2 e^x}{(e^x+1)^2}, \qquad x \equiv \frac{\varepsilon-\varepsilon_F}{\tau}$$

Since $\tau \ll \varepsilon_F$, the integral equals $\pi^2/3$, the specific heat is

$$\boxed{C_{el} = \frac{\pi^2}{3} D(\varepsilon_F)\, \tau}$$ ← linear τ dependence

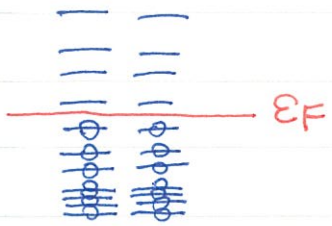
Now we can try to relate $D(\varepsilon_F)$ with $\varepsilon_F = \tau_F$. From previous calculations,

$$D(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \sqrt{\varepsilon} \quad \text{and} \quad N = \int_0^{\varepsilon_F} d\varepsilon\, D(\varepsilon)$$

$$\rightarrow \quad N = \underbrace{\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}}_{D(\varepsilon_F)/\sqrt{\varepsilon_F}} \int_0^{\varepsilon_F} d\varepsilon \sqrt{\varepsilon} = \frac{D(\varepsilon_F)}{\sqrt{\varepsilon_F}} \cdot \frac{2}{3} \varepsilon_F^{\frac{3}{2}}$$

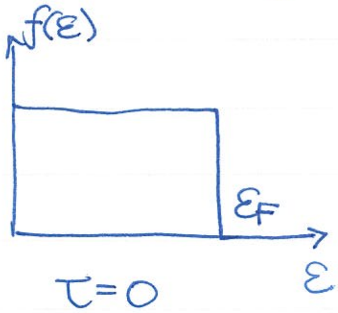
Therefore, we arrive at the very important relation:

$$\boxed{D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}}$$ ← density of states (along ε axis).

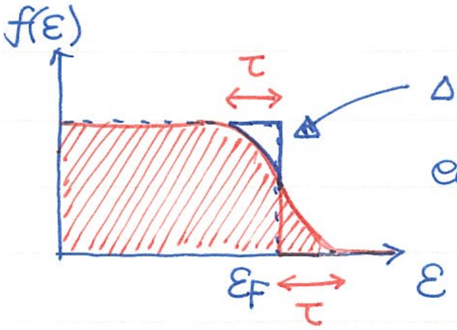


Although the density of states $D(\varepsilon)$ is not uniform, a rough estimate for $D(\varepsilon_F) \approx \frac{N}{\varepsilon_F}$. This is not bad at all when compared with the exact results derived in above.

A pictorial way to understand $U_{el} \sim \tau$:



$\tau = 0$



$\Delta N \sim N \cdot \left(\frac{\tau}{\varepsilon_F}\right) \cdot \frac{1}{4}$

energy shift $\sim \tau$

Therefore, $\Delta U \sim \frac{N}{4}\left(\frac{\tau}{\varepsilon_F}\right) \cdot \tau \approx \frac{N}{4\varepsilon_F}\tau^2$

$$C_{el} = \frac{dU}{d\tau} \sim \frac{1}{2}\frac{N\tau}{\varepsilon_F} \quad \left[\text{compare with } \frac{\pi^2}{2}\frac{N\tau}{\varepsilon_F}\right]$$

Under thermal fluctuations, only a small portion of electrons (roughly $\tau/\varepsilon_F \approx 1/100$ at room temperature) remains active. The Fermi sea is quite robust and not affected by thermal fluctuations. Why? Protected by Pauli's exclusion principle.

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