HH0004 Quantum Statistics

In order to appreciate quantum statistics, one needs to know some basic quantum physics. Unlike classical mechanics, one needs "wave function" to describe a particle.

$\Psi(x)$

Wave function.

$\boxed{|\Psi(x)|^2 \text{ is the probability density to find the particle at } x}$

Probability Conservation $\longrightarrow$ $\displaystyle\int_{-\infty}^{+\infty} |\Psi(x)|^2 dx = 1.$

What about two particles? Assume there's no interaction between them. The wave function of the system should be

$L(x_2)$

$R(x_1)$

$$\boxed{\Psi(x_1, x_2) = R(x_1)\, L(x_2)}$$

Does this make sense?

Note that $|\Psi(x_1,x_2)|^2 = |R(x_1)|^2 \cdot |L(x_2)|^2$ is the probability density to find $\cdots\rightarrow$ [particle 1 at x_1 & particle 2 at x_2] This seems reasonable ☺

But, what happens if these two particles are identical? It's reasonable to request $|\Psi(x_1,x_2)|^2$ invariant when $x_1 \leftrightarrow x_2$. In addition, exchanging the particles twice should give back the same wave function.

$$\longrightarrow \quad \boxed{\Psi(x_1, x_2) = \pm\, \Psi(x_2, x_1)}$$

$+$ Boson.

$-$ Fermion.

Going back to the example:

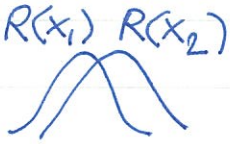
identical particles.

Therefore, the wave function for the two-particle system is

$$\boxed{\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[R(x_1)L(x_2) \pm L(x_1)R(x_2)\right]}$$

$\frac{1}{\sqrt{2}}$ factor: $\iint |\Psi|^2 dx_1 dx_2 = 1.$ Probably do not need to go into details....

(I) Pauli exclusion principle:



$$\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}\left[R(x_1)R(x_2) - R(x_1)R(x_2)\right]$$
$$= 0 \text{ !!}$$

" ✗ " Two fermions can't occupy the same orbital.

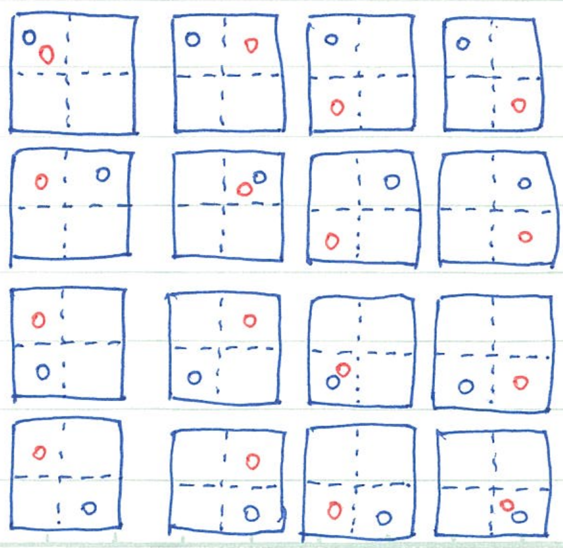
In fact, it is easy to see that

$$\Psi(x, x) = -\Psi(x, x) = 0$$

Fermions hate each other.

(II) Bunching versus exclusion

Consider 2 identical particles in a box. For simplicity, assume there are only 4 orbitals.



<u>classical</u> $g = 16$ states

$$P_2 = \frac{4}{16} = \frac{1}{4} = \underline{25\%}$$

find two particles together.

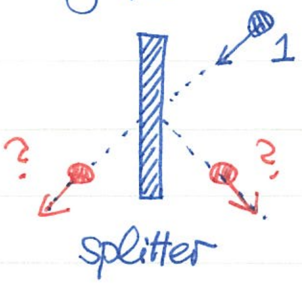
boson Because particles are identical, $g = 10$ ← reduced!

$$P_2 = \frac{4}{10} = \frac{2}{5} = \underline{\underline{40\%}}$$ ← 15% enhanced bunching behavior.

fermion The multiplicity is further reduced ....

$$g = 6 \longrightarrow P_2 = \frac{0}{6} = \underline{\underline{0}}$$ Pauli exclusion principle ☺

Let's turn to another interesting example. A particle coming from an orbital R hits the splitter.



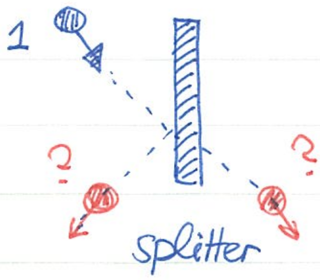
$$\boxed{\begin{aligned} &R(1) \longrightarrow C_R R(1) + C_L L(1) \\ &\text{with } |C_R|^2 + |C_L|^2 = 1 \end{aligned}}$$

Probability to R/L is $|C_R|^2 / |C_L|^2$. Thus, they sum up to unity. If the splitting is symmetric, $|C_R|^2 = |C_L|^2 = \frac{1}{2}$. For convenice, we can choose

$$\boxed{R(1) \longrightarrow \frac{1}{\sqrt{2}} R(1) + \frac{1}{\sqrt{2}} L(1)}$$

Similarly, one can consider the particle coming from the orbital L. From the standard scattering theory,

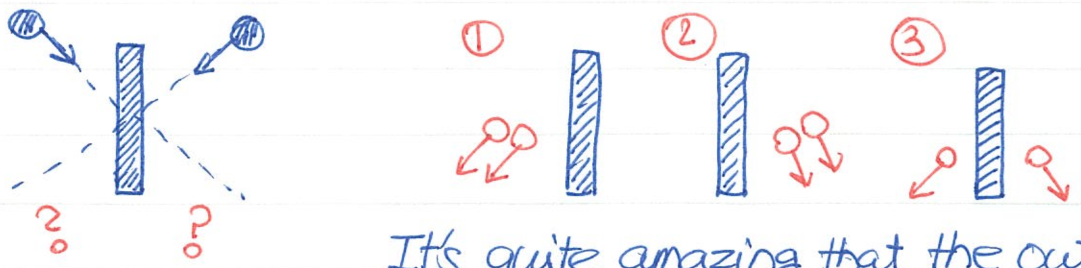
$$\boxed{L(1) \longrightarrow -\frac{1}{\sqrt{2}} R(1) + \frac{1}{\sqrt{2}} L(1)}$$



That is to say, the scattering matrix is

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$ ← explain when asked ☺

Suppose two identical particles incident from both sides.
What happens? Three possible outcomes.



It's quite amazing that the outcome depends on quantum statistics.

initial state $\boxed{\Psi_{in} = \frac{1}{\sqrt{2}}\left[R(1)L(2) \pm L(1)R(2)\right]}$

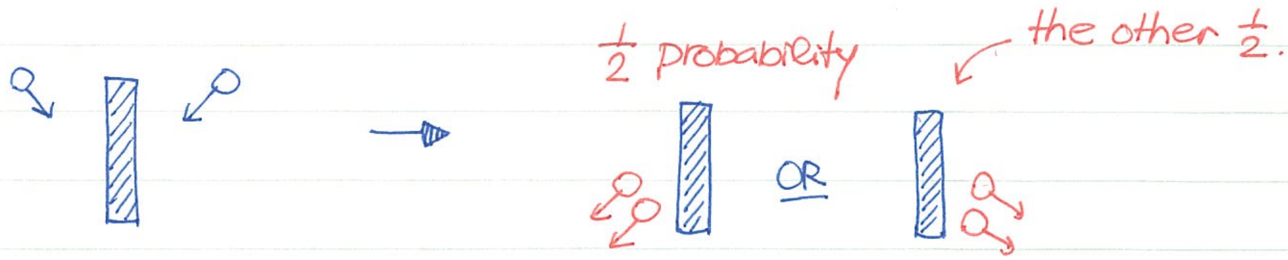
After scattering, the final state changes ....

final state
$$\Psi_{out} = \frac{1}{\sqrt{2}}\left\{\left[\frac{1}{\sqrt{2}}R(1)+\frac{1}{\sqrt{2}}L(1)\right]\left[\frac{1}{\sqrt{2}}L(2)-\frac{1}{\sqrt{2}}R(2)\right] \pm \left[\frac{1}{\sqrt{2}}L(1)-\frac{1}{\sqrt{2}}R(1)\right]\left[\frac{1}{\sqrt{2}}R(2)+\frac{1}{\sqrt{2}}L(2)\right]\right\}$$

$$\boxed{\begin{aligned}\Psi_{out} &= \frac{1}{2\sqrt{2}}\left[L(1)L(2)-R(1)R(2)+R(1)L(2)-L(1)R(2)\right] \\ &\pm \frac{1}{2\sqrt{2}}\left[L(1)L(2)-R(1)R(2)-R(1)L(2)+L(1)R(2)\right]\end{aligned}}$$

For boson, take the plus sign.

$$\Psi_{out} = \frac{1}{\sqrt{2}}\left[L(1)L(2)-R(1)R(2)\right]$$



The boson bunching effect is dramatic here !!

For fermion, take the minus sign

$$\Psi_{out} = \frac{1}{\sqrt{2}}\left[R(1)L(2) - L(1)R(2)\right]$$

the same as Ψ_{in} ☺

The outcome is definite because fermions do not like the other two outcomes.....

The outgoing state is the same as the incoming state — one particle on each side with antisymmetrize wave function. This scattering setup demonstrates the importance of quantum statistics.

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