On the Stable Homotopy Type of Certain Thom Spectra

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ABSTARCT

It is well-known that $H(\mathbb{Z}/2)$ is a Thom spectrum, observed by Mark Mahowald. In this paper, our main purpose is to give a generalization of this. The generalization is observed by Dung-Yung Yan. Our proof will follow closely a short proof by Dung-Yung Yan in [7].

CONTENTS

1 Introduction

The main result of this paper is to give a generalization of Mahowald's striking observation that the mod 2 Eilenberg-MacLane spectrum is a Thom spectrum. This generalization is observed by Dung-Yung Yan in [7].

First, we recall the definition of Thom spectra as follow.

Let $f : L \longrightarrow BO$ be an H-map. Choose a filtration $\{L_n\}$ of L such that $f(L_n) \subseteq BO(n)$. Let $f_n = f\mid_{L_n}$. Consider a universal n-plane bundle γ_n over $BO(n)$. Set $\overline{\gamma}_n = (f_n)^*(\gamma_n)$. Then we define $Th(f)_n = E(\overline{\gamma}_n)/A$, where A is the subset of $E(\overline{\gamma}_n)$ consisting of those vectors of length at least 1 in each fibre. We call it a Thom space with respect to $\overline{\gamma}_n$.

Because we have the commutative diagram

$$\begin{tikzcd} \overline{\gamma}_n \oplus \varepsilon^1 \arrow[r] \arrow[d] & \overline{\gamma}_{n+1} \arrow[d] \\ L_n \arrow[r, "\subseteq"] & L_{n+1}, \end{tikzcd}$$

where ε^1 is a trivial 1-plane bundle over L_n, we have the structure map

$$\epsilon_n : S^1 \wedge Th(f)_n \longrightarrow Th(f)_{n+1}.$$

Hence, we obtain the Thom spectrum, $Th(f)$.

In [4], Mahowald shows that the Thom spectrum, $Th(f)$, is a ring spectrum, with the ring structure map induced by the following commutative diagram:

$$\begin{tikzcd} L \times L \arrow[r, "\mu_L"] \arrow[d, "f\times f"'] & L \arrow[d, "f"] \\ BO \times BO \arrow[r, "\mu_{BO}"] & BO, \end{tikzcd}$$

where μ_L and μ_{BO} are multiplications for H-spaces L and BO, respectively.

Furthermore, if L is a commutative H-space and f is a morphism of commutative H-spaces, then $Th(f)$ is a commutative ring spectrum.

According to [1], we can see that $Th(f)$ is (-1)-connected and $\pi_0(Th(f))$ is either $\mathbb{Z}$ or $\mathbb{Z}/2$. As f is non-orientable, i.e., $f^*(\omega_1) \neq 0$, $\pi_0(Th(f)) = \mathbb{Z}/2$. Otherwise, $\pi_0(Th(f)) = \mathbb{Z}$.

For example, if we take $L = BO$, which is the infinite loop space, and f is the identity mapping of BO, we obtain the Thom spectrum MO.

Note that the homology $H_*(X)$ always means with $(\mathbb{Z}/2)$-coefficient in this paper. Now we state our main results as follow.

Theorem 1.1. *Given a connected CW-complex L. Suppose $f : L \longrightarrow BO$ is a non-orientable double loop map. Then $H_*(Th(f))$ is an extended comodule over the mod 2 dual Steenrod algebra $A_* = H_*(H(\mathbb{Z}/2))$, i.e., $H_*(Th(f)) \cong A_* \otimes_{\mathbb{Z}/2} C$, with the following comodule structure:*

$$A_* \otimes_{\mathbb{Z}/2} C \xrightarrow{\psi\otimes id} (A_* \otimes_{\mathbb{Z}/2} A_*) \otimes_{\mathbb{Z}/2} C \xrightarrow{\cong} A_* \otimes_{\mathbb{Z}/2} (A_* \otimes_{\mathbb{Z}/2} C),$$

where $\psi : A_* \longrightarrow A_* \otimes A_*$ is the co-product on A_*.

Theorem 1.2. *Given a connected CW-complex L. Suppose $f : L \longrightarrow BO$ is a non-orientable double loop map. Then the Thom spectrum $Th(f)$ can be split as a wedge of suspensions of Eilenberg-MacLane spectra $H(\mathbb{Z}/2)$.*

Corollary 1.3 (Thom, [2]). *$H_*(MO)$ is an extended A_*-comodule. Hence MO can be stably split as the wedge of suspensions of Eilenberg-MacLane spectra $H(\mathbb{Z}/2)$.*

Corollary 1.4. *Consider a Thom spectrum $Th(f_1)$ induced by the following fibration:*

$$U/O \xrightarrow{f_1} BO \longrightarrow BU .$$

Then $H_*(Th(f_1))$ is an extended comodule, and so $Th(f_1)$ can be stably split as the wedge of suspensions of Eilenberg-MacLane spectra $H(\mathbb{Z}/2)$.

Corollary 1.5. *Consider a Thom spectrum $Th(f_2)$ induced by the following fibration:*

$$Sp/O \xrightarrow{f_2} BO \longrightarrow BSp .$$

Then $H_*(Th(f_2))$ is an extended comodule, and so $Th(f_2)$ can be stably split as the wedge of suspensions of Eilenberg-MacLane spectra $H(\mathbb{Z}/2)$.

2 Proof of Our Theorems

In order to prove Theorem 1.1 and Theorem 1.2, we will apply two theorems; one is Mahowald's Theorem due to [3], and the other is the comodule structure theorem due to [5]. Now, we recall them as follow.

Let η represent the generator of $\pi_1(BO) = \mathbb{Z}/2$. Since BO is a double loop space, we have a map

$$g : \Omega^2 S^3 = \Omega^2\Sigma^2 S^1 \xrightarrow{\Omega^2\Sigma^2\eta} \Omega^2\Sigma^2 BO \longrightarrow BO.$$

Theorem 2.1 (Mahowald, [3]). *Thom spectrum $Th(g)$ is the Eilenberg-MacLane spectrum $H(\mathbb{Z}/2)$.*

Remark 2.2. In [6], Priddy shows that the composite map

$$H(\mathbb{Z}/2) \xrightarrow{\overline{g}} MO \xrightarrow{\alpha} H(\mathbb{Z}/2)$$

is a homotopic equivalence, where the first map $\overline{g}$ is the map of Thom spectra induced by g and the second map α is the Thom class which represents the generator of $H^0(MO) = \mathbb{Z}/2$. Hence, there exists a map $\lambda : MO \longrightarrow H(\mathbb{Z}/2)$ such that the following composite map

$$H(\mathbb{Z}/2) \xrightarrow{\overline{g}} MO \xrightarrow{\lambda} H(\mathbb{Z}/2)$$

is homotopic to the identity map. $\square$

Next, we state the comodule structure theorem. Before we state this theorem, we recall the definition about cotensor product.

Definition 2.3 (Cotensor product). *Given a Hopf algebra A over K. If M is a right A-comodule, with the right coaction $\Delta_M : M \longrightarrow M \otimes A$, and N is a left A-comodule, with the left coaction $\Delta_N : N \longrightarrow A \otimes N$, then the cotensor product of M and N, denoted by $M \Box_A N$, is the kernel of $[\Delta_M \otimes id_N - id_M \otimes \Delta_N] : M \otimes N \longrightarrow M \otimes A \otimes N$.*

Theorem 2.4 (Milnor and Moore, [5]). *Let A be a commutative connected Hopf algebra over a field K, i.e., $A_0 \cong K$. Let B be a connected left A-comodule algebra and $C = K \Box_A B$, the cotensor product of K and B. If there is a surjective homomorphism $g : B \longrightarrow A$ of left A-comodule algebras, then B is isomorphic to $A \otimes_K C$ simultaneously as a left A-comodule and a right C-module.*

Before we start to prove our main theorems, we first prove the following lemma.

Lemma 2.5. *Given two spectra X and Y of finite type. Suppose that there exists a map $f : X \longrightarrow Y$, such that f_* is an isomorphism from $H_*(X, \mathbb{Z}/2)$ onto $H_*(Y, \mathbb{Z}/2)$, then $f_{(2)}$ is a homotopy equivalence from $X_{(2)}$ to $Y_{(2)}$, which is induced by f.*

Proof. Consider a cofibration $X \xrightarrow{f} Y \xrightarrow{i} C_f$. Then we have a long exact sequence

$$\cdots \longrightarrow H_*(X, \mathbb{Z}/2) \xrightarrow{f_*} H_*(Y, \mathbb{Z}/2) \xrightarrow{i_*} H(C_f, \mathbb{Z}/2) \longrightarrow \cdots.$$

Since $H_*(X, \mathbb{Z}/2) \cong H_*(Y, \mathbb{Z}/2)$, $H_*(C_f, \mathbb{Z}/2) \cong 0$. It is following that $H_*((C_f)_{(2)}, \mathbb{Z}/2) \cong 0$.

By the Adams spectral sequence, we have $\pi_*((C_f)_{(2)}) \cong 0$. Because

$$X_{(2)} \xrightarrow{f_{(2)}} Y_{(2)} \xrightarrow{i_{(2)}} (C_f)_{(2)} \simeq C_{f_{(2)}}$$

is also a cofibration, it is a fibration in the stable category. This implies that we have a long exact sequence

$$\cdots \longrightarrow \pi_*(X_{(2)}) \xrightarrow{(f_{(2)})_\#} \pi_*(Y_{(2)}) \xrightarrow{(i_{(2)})_\#} \pi_*((C_f)_{(2)}) \longrightarrow \cdots,$$

and so there is a short exact sequence

$$0 \cong \pi_{*+1}((C_f)_{(2)}) \longrightarrow \pi_*(X_{(2)}) \xrightarrow{(f_{(2)})_\#} \pi_*(Y_{(2)}) \xrightarrow{(i_{(2)})_\#} \pi_*((C_f)_{(2)}) \cong 0.$$

Thus, $(f_{(2)})_\#$ is an isomorphism from $\pi_*(X_{(2)})$ onto $\pi_*(Y_{(2)})$. That is to say, $f_{(2)}$ is a homotopy equivalence from $X_{(2)}$ to $Y_{(2)}$, and then we complete the proof. $\square$

Now we prove our main theorems.

Proof of Theorem 1.1:

First, let $L = \Omega^2 X$, for some space X. Since $f^*(\omega_1) \neq 0$ and the following diagram commutes

$$\begin{tikzcd}
\pi_1(\Omega^2 X) \arrow[r, "f_\#"] \arrow[d, "onto"'] & \pi_1(BO) \arrow[d, "onto"] \\
H_1(\Omega^2 X) \arrow[r, "f_*"] & H_1(BO),
\end{tikzcd}$$

there is a map $\iota : S^1 \longrightarrow \Omega^2 X$ such that the composite map

$$S^1 \xrightarrow{\iota} \Omega^2 X \xrightarrow{f} BO$$

is homotopic to the map η, representing the generator of $\pi_1(BO)$.

Since f is a double loop map, this implies that the following diagram

$$\begin{tikzcd}
\Omega^2 S^3 \arrow[rr, "g"] \arrow[dr, "h"'] & & BO \\
& \Omega^2 X \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes up to homotopy, where $h : \Omega^2 S^3 \longrightarrow \Omega^2 X$ is induced by ι.

By Theorem 2.1, we have the commutative diagram:

$$\begin{tikzcd}
H(\mathbb{Z}/2) \arrow[rr, "\overline{g}"'] \arrow[dr, "\overline{h}"'] \arrow[rrr, bend left, "\simeq id"] & & MO \arrow[r, "\lambda"'] & H(\mathbb{Z}/2), \\
& Th(f) \arrow[ur, "\overline{f}"'] & &
\end{tikzcd}$$

where $\overline{h}$ and $\overline{f}$ are thomfied by h and f, respectively.

This induces the commutative diagram

$$\begin{tikzcd}
A_* \arrow[rr, "\overline{g}_*"] \arrow[dr, "\overline{h}_*"'] & \arrow[d, phantom, "\circlearrowleft"] & H_*(MO) \arrow[r, "\lambda_*"] & A_*, \\
& H_*(Th(f)) \arrow[ur, "\overline{f}_*"'] & &
\end{tikzcd}$$

where $\lambda_* \circ \overline{g}_* = id$.

So we have a map $\Phi = \lambda_* \circ \overline{f}_* : H_*(Th(f)) \longrightarrow A_*$, which is a surjective homomorphism of left comodule algebras. Set $C = (\mathbb{Z}/2)\square_{A_*}(H_*(Th(f)))$. By Theorem 2.4, $H_*(Th(f))$ is isomorphic to $A_* \otimes_{\mathbb{Z}/2} C$, as a left comodule algebra. That is to say, $H_*(Th(f))$ is an extended A_*-comodule. $\square$

Proof of Theorem 1.2:

In Theorem 1.1, we have shown the algebraic isomorphism. Now we only have to construct the topological map such that it induces an isomorphism of mod 2 homology and then we complete the proof.

First, we claim that every element of C is a stable sphere, i.e., C is equal to the image of Hurewicz homomorphism which maps $\pi_*(Th(f))$ to $H_*(Th(f))$.

Since C is the set of consisting of coaction primitive elements in the A_*- comodule $H_*(Th(f))$, we see that the image of Hurewicz homomorphism is contained in C. Next, we want to show that E_2-term for the Adams spectral sequence collapses to E_∞-term.

It is easy to check that

$$\begin{aligned} E_2^{s,t} &= Ext_{A_*}^{s,t}(\mathbb{Z}/2, H_*(Th(f))) \\ &\cong Ext_{A_*}^{s,t}(\mathbb{Z}/2, A_* \otimes_{\mathbb{Z}/2} C) \\ &= Ext_{A_*}^{s,t}(\mathbb{Z}/2, A_*) \otimes_{\mathbb{Z}/2} C \\ &= \begin{cases} C & , if\ s = 0 \\ 0 & , if\ s \neq 0 \end{cases} \end{aligned}$$

This implies $E_2^{*,*} \cong E_3^{*,*} \cong \cdots \cong E_\infty^{*,*}$, and so each generator of C is a permanent cycle.

Moreover, since $\pi_0(Th(f)) = \mathbb{Z}/2$, $\pi_*(Th(f))$ has characteristic 2, and thus $Th(f)$ is 2-local. This implies that the Adams spectral sequence $E_2^{*,*}$ converges to $\pi_*(Th(f))$. It is following that each generator of C is stably spherical.

Next, let c be any generator of C. Then there exists an essential map g_α from S^α to $Th(f)$, such that $(g_\alpha)_*(i_\alpha) = c$, where i_α is the generator of $H_\alpha(S^\alpha)$. Let γ be the following composite map:

$$\bigvee_\alpha S^\alpha \xrightarrow{\bigvee_\alpha g_\alpha} \bigvee_\alpha Th(f) \xrightarrow{\nabla} Th(f),$$

where the second map ∇ is the folding map. Note that γ_* is an isomorphism from $H_*(\bigvee_\alpha S^\alpha)$ onto C, by the construction of g_α. Thus, we have the following composite map θ:

$$H(\mathbb{Z}/2) \wedge (\bigvee_\alpha S^\alpha) \xrightarrow{id \wedge \gamma} H(\mathbb{Z}/2) \wedge Th(f) \xrightarrow{\overline{h} \wedge id} Th(f) \wedge Th(f) \xrightarrow{\mu} Th(f),$$

where $\mu : Th(f) \wedge Th(f) \longrightarrow Th(f)$ is the structure map of $Th(f)$.

In order to prove that θ_* is an isomorphism, we need to describe two homomorphisms

$$\overline{h}_* : A_* \longrightarrow H_*(Th(f)) \text{ and } \gamma_* : H_*(\bigvee_\alpha S^\alpha) \longrightarrow H_*(Th(f)).$$

Clearly, γ_* maps $H_*(\bigvee_\alpha S^\alpha)$ onto C. By Theorem 2.1, we see that $\overline{h}_*$ is an isomorphism from A_* onto the part A_* of $H_*(Th(f)) \cong A_* \otimes_{\mathbb{Z}/2} C$, and so θ_* is an isomorphism.

By Lemma 2.5, we see that

$$\theta_\# : \pi_*(H(\mathbb{Z}/2) \wedge (\bigvee_\alpha S^\alpha)) \longrightarrow \pi_*(Th(f))$$

is an isomorphism. By Whitehead Theorem, we have

$$Th(f) \simeq H(\mathbb{Z}/2) \wedge (\bigvee_\alpha S^\alpha) = \bigvee_\alpha \Sigma^\alpha H(\mathbb{Z}/2).$$

$\square$

Proof of Corollary 1.4:

It is well-known that U/O and BO are infinite loop spaces, and $f_1 : U/O \longrightarrow BO$ is an infinite double loop map. It is enough to show that f_1 is non-orientable. Moreover, we know that

$$U/O \xrightarrow{f_1} BO \longrightarrow BU$$

is a fibration. Hence, we have an exact sequence

$$\pi_1(U/O) \xrightarrow{(f_1)_\#} \pi_1(BO) \longrightarrow \pi_1(BU) = 0.$$

That is to say, $(f_1)_\# : \pi_1(U/O) \longrightarrow \pi(BO)$ is surjective. Hence, $(f_1)^*(w_1) \neq 0$. By Theorem 1.1 and Theorem 1.2, we see that $Th(f_1)$ can be stably split as the wedge of suspensions of Eilenberg-MacLane spectra $H(\mathbb{Z}/2)$. $\square$

Proof of Corollary 1.5:

Similar to Corollary 1.4 $\square$

References

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