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在基本群上的一個敏銳觀察

A sharp observation of fundamental group

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Abstract

We will give an extremely new idea to approach the homotopy group of sphere.

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1 Introduction

Recall that $\pi_1(S^1)$ is isomorphic to integer $\mathbb{Z}$[1][4]. Tranditionally, we may use some concepts of covering space and lifting lemma to obtain this result[3]. These methods are more biased in favor of the topology view. Unlike the previous, we will use a new idea to calculate $\pi_1(S^1)$ in this paper.

Recall that 1-sphere $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, we choose $(1,0)$ be the base point of S^1. We can define that counterclockwise be positive direction on S^1, and clockwise be negative direction on S^1. So there is an orientation on S^1.

From now on, when we talk about a path f and a loop f, it means $f : [0,1] \to S^1$ with $f(0) = (1,0)$ and $f : [0,1] \to S^1$ with $f(0) = f(1) = (1,0)$, respectively. Moreover, we always write $f(s) = (x(s), y(s))$ for all $s \in [0,1]$.

Given a path f, we will define the cumulate length of f from 0 to s, where $s \in (0,1]$. Given s_1 and s_2 in $[0,1]$, where $s_1 < s_2$. If the path $f : [s_1, s_2] \to S^1$ moves along counterclockwise direction, then we define the cumulate length of f from s_1 to s_2 is $\int_{s_1}^{s_2} \sqrt{(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2}ds$. If the path $f : [s_1, s_2] \to S^1$ moves along clockwise direction, then we define the cumulate length of f from s_1 to s_2 is $-\int_{s_1}^{s_2} \sqrt{(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2}ds$. If the path $f : [s_1, s_2] \to S^1$ stays at $f(s_1)$, then we define the cumulate length of f from s_1 to s_2 is 0. So given $s \in (0,1]$, we can given a partition of $[0,s]$ such that for each subinterval $[s_i, s_{i+1}]$ of $[0,s]$, the path $f : [s_i, s_{i+1}] \to S^1$ moves along either counterclockwise or clockwise direction, for each i. We define $\bar{f}(s)$ be the sum of each cumulate length of f from s_i to s_{i+1}, and called it the cumulate length of f from 0 to s. Use the similar concept, we can also define the cumulate length of f from a to b, where $0 \le a < b \le 1$. Note that the cumulate length of f from 0 to 0 should be 0. That is, $\bar{f}(0) = 0$.

For example, define a path f by

$$f(s) = \begin{cases} (\cos(4\pi s), \sin(4\pi s)) & \text{, if } s \in [0, \frac{1}{4}], \\ (-1, 0) & \text{, if } s \in [\frac{1}{4}, \frac{1}{2}], \\ (\cos(2\pi s), -\sin(2\pi s)) & \text{, if } s \in [\frac{1}{2}, \frac{3}{4}], \\ (\cos(14\pi s), \sin(14\pi s)) & \text{, if } s \in [\frac{3}{4}, 1], \end{cases}$$

then $\bar{f}(\frac{1}{4}) = \pi$, $\bar{f}(\frac{1}{2}) = \pi$, $\bar{f}(\frac{3}{4}) = \frac{\pi}{2}$, $\bar{f}(\frac{6}{7}) = 2\pi$, and $\bar{f}(1) = 4\pi$.

However, for any path f, we need to prove that $\bar{f}$ is well-defined. That is, we have to claim that $\bar{f}(s) \neq \pm\infty$ for all $s \in [0, 1]$. We will prove it in section 2. So for any path f, it induces a function $\bar{f} : [0, 1] \to \mathbb{R}$ such that the meaning of $\bar{f}(s)$ is the cumulate length of f from 0 to s, for each $s \in [0, 1]$. Moreover, $\bar{f}$ is continuous. It will be proved in Theorem 2.1.

Obviously, if f is a loop, there is an integer n such that $\bar{f}(1) = 2n\pi$. That is, for each loop f, it induces an integer $\frac{1}{2\pi}\bar{f}(1)$. In Theorem 2.2, we describe that given two loops f and g, then $f \simeq g$ rel $\{0, 1\}$ if and only if $\bar{f}(1) = \bar{g}(1)$, where the notation $f \simeq g$ rel $\{0, 1\}$ is defined by Definition 2.2.5 in p.27 of [2]. This theorem will be proved in section 3. So we have a well-defined injective map $\chi : \pi_1(S^1) \to \mathbb{Z}$ by

$$\chi([f]) = \frac{1}{2\pi}\bar{f}(1).$$

In Theorem 2.3, we will prove that χ is also surjective and a homomorphism. Hence χ is an isomorphism. Therefore, $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$, and we complete our main purpose in this paper.

2 Prove that $\pi_1(S^1) \cong \mathbb{Z}$

Given a path f, we have had the concept of the cumulate length of f from 0 to s in section 1. However, we need to prove that $\bar{f}$ is well-defined. That is, we have to claim that $\bar{f}(s) \neq \pm\infty$ for all $s \in [0,1]$. In order to prove this result, we need a lemma.

Lemma 2.1. *Let f be a path. Then for any $s \in (0,1]$, there is no sequence $\{s_i\}_{i=1}^{\infty}$, $0 \leq s_i \leq s$, with the property $f([s_i, s_{i+1}]) = S^1$ and $f(s_i) = c$, where $i = 1, 2, 3, \cdots$, and c is a fixed point in S^1.*

Proof. Suppose we had a sequence $\{s_i\}_{i=1}^{\infty}$ such that $0 \leq s_i \leq s$, $f([s_i, s_{i+1}]) = S^1$, and $f(s_i) = c$, where $i = 1, 2, 3, \cdots$. Since $\{s_i\}_{i=1}^{\infty}$ would be bounded above, let $a = \sup\{s_i\}_{i=1}^{\infty}$. Then since f is continuous, $f(a) = c \in S^1$. Pick up $\tilde{c} \in S^1$ with $\tilde{c} \neq c$. Then since $f([s_i, s_{i+1}]) = S^1$, for each i, there would be α_i, $s_i \leq \alpha_i \leq s_{i+1}$, such that $f(\alpha_i) = \tilde{c}$. Since $a = \sup\{s_i\}_{i=1}^{\infty}$, $a = \sup\{\alpha_i\}_{i=1}^{\infty}$. There would be a subsequence $\{\alpha_j\}_{j=1}^{\infty}$ of $\{\alpha_i\}_{i=1}^{\infty}$ such that $\lim\limits_{j\to\infty} \alpha_j = a$. It implies

$$c = f(a) = f(\lim_{j\to\infty} \alpha_j) = \lim_{j\to\infty} f(\alpha_j) = \tilde{c}.$$

So we get a contradiction, and complete this proof. □

We are ready to write the form of $\bar{f}$ and to prove that $\bar{f}$ is well-defined. First we assign an integer n_s for each $s \in [0,1]$. If $s = 0$, then define $n_s = 0$. If $s \in (0,1]$, then let $S = \{x \mid x \leq s \text{ and } f(x) = (1,0)\}$. Since S is bounded above, let $m = \sup S$. Then since f is continuous, $f(m) = (1,0)$. If $m = 0$, then define $n_s = 0$. If $m \in (0, s]$, then the cumulate length of f from 0 to m may be $2n\pi$ or $\pm\infty$, where $n \in \mathbb{Z}$. That is, the path $f : [0,m] \to S^1$ may wrap completely $|n|$ or infinitely many circles around S^1. We claim that the case of infinitely many circles is impossible. Suppose the path $f : [0,m] \to S^1$ wraped completely infinitely many circles around S^1. Then there would be a sequence $\{s_i\}_{i=1}^{\infty}$, $0 \leq s_i \leq m$, with the property $f([s_i, s_{i+1}]) = S^1$ and $f(s_i) = (1,0)$, where $i = 1, 2, 3, \cdots$. It contradicts to Lemma 2.1. Hence the only possibility is there exsits an integer n such that the cumulate length of f from 0 to m is $2n\pi$. Let $n_s = n$. So for each $s \in [0,1]$, we get an integer n_s.

Remark 2.1. *In the following, we keep in mind $m = \sup S$, where S is defined as above, $f(m) = (1,0)$ and $f((m,s])$ does not contain $(1,0)$.*

We next define a map $l : [0,1] \to (-2\pi, 2\pi)$ as follows. If $f(s) = (1,0)$, then define $l(s) = 0$. If $f(s) \neq (1,0)$, then define

$$l(s) = (-1)^{\lambda(s)}\pi + (-1)^{\mu(s)} \int_{-1}^{x(s)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Note that

$$\mu(s) = \begin{cases} 0 & \text{, if } y(s) < 0, \\ 1 & \text{, if } y(s) > 0 \text{ or } f(s) = (-1,0), \end{cases}$$

and $y(x) = \sqrt{1-x^2}$. Moreover, $\lambda(s) = 0$, if the path $f : [m,s] \to S^1$ is moving away from $f(m) = (1,0)$ along counterclockwise direction; $\lambda(s) = 1$, if the path $f : [m,s] \to S^1$ is moving away from $f(m) = (1,0)$ along clockwise direction. So the meaning of $l(s)$ is the cumulate length of f from m to s.

Therefore, $l(s) + 2n_s\pi$ is the cumulate length of f from 0 to s, for some $n_s \in \mathbb{Z}$. That is,

$$\bar{f}(s) = l(s) + 2n_s\pi, \tag{1}$$

for all $s \in [0,1]$. Since $n_s \in \mathbb{Z}$ and $-2\pi < l(s) < 2\pi$, $\bar{f}(s) \in \mathbb{R}$, for all $s \in [0,1]$. That is, $\bar{f}$ is a function from $[0,1]$ into $\mathbb{R}$.

We are on the position to describe our main theorem.

Theorem 2.1. *Let f be a path. Then it induces a continuous function $\bar{f} : [0,1] \to \mathbb{R}$ which is defined by equation (1), and the meaning of $\bar{f}(s)$ is the cumulate length of f from 0 to s, for each $s \in [0,1]$.*

Proof. According to the above argument, we have proved that $\bar{f}$ is well-defined and the meaning of $\bar{f}(s)$ is the cumulate length of f from 0 to s, for each $s \in [0,1]$. It remains to prove that $\bar{f}$ is continuous.

Given $s_0 \in [0,1]$. Then $f(s_0)$ may be equal to $(1,0)$ or not. Suppose $f(s_0) \neq (1,0)$. Then for any s is close to s_0, it is obviously $n_s = n_{s_0}$ and $l(s)$ is close to $l(s_0)$. It implies that $\bar{f}(s)$ is close to $\bar{f}(s_0)$. Hence $\bar{f}$ is continuous at s_0 for $f(s_0) \neq (1,0)$. Suppose $f(s_0) = (1,0)$. Note that $l(s_0) = 0$. Then for any s is close to s_0, it is obviously either $n_s = n_{s_0}$ and $l(s)$ is close to $l(s_0) = 0$ or $n_s = n_{s_0} \mp 1$ and $l(s)$ is close to $l(s_0) \pm 2\pi = \pm 2\pi$. It implies that $\bar{f}(s)$ is close to $\bar{f}(s_0)$. Hence $\bar{f}$ is continuous at s_0 for $f(s_0) = (1,0)$. To combine above two cases, we get $\bar{f}$ is continuous at s_0. Since s_0 is arbitrary in $[0,1]$, $\bar{f}$ is a continuous function. So we complete this proof. $\square$

Remark 2.2. *Let f be a path. Since $f(s) \in S^1$ for each $s \in [0,1]$, the cumulate length is equal to the cumulate angle of f from 0 to s. That is,*

$$(\cos(\bar{f}(s)), \sin(\bar{f}(s))) = f(s),$$

for all $s \in [0,1]$.

Remark 2.3. *If f is a loop, then $\bar{f}(1) = 2n_1\pi$ for some $n_1 \in \mathbb{Z}$ by equation (1). Therefore, for each loop f, it induces an integer $\frac{1}{2\pi}\bar{f}(1)$.*

Before to prove that $\pi_1(S^1) \cong \mathbb{Z}$, we need a theorem. It will be proved in section 3.

Theorem 2.2. *Let f and g be loops. Then $f \simeq g$ rel $\{0,1\}$ if and only if $\bar{f}(1) = \bar{g}(1)$.*

We are on the position to prove that $\pi_1(S^1) \cong \mathbb{Z}$.

Theorem 2.3. *Define a map $\chi : \pi_1(S^1) \to \mathbb{Z}$ by*

$$\chi([f]) = \frac{1}{2\pi}\bar{f}(1),$$

for any loop f. Then χ is an isomorphism. Therefore, $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$.

Proof. First by Remark 2.3, the range of χ is contained in $\mathbb{Z}$. Secondly we have to prove that χ is well-defined and injective. Given two loops f and g. Assume that $f \simeq g$ rel $\{0,1\}$. Then by Theorem 2.2, $\bar{f}(1) = \bar{g}(1)$. So χ is well-defined. On the other hand, assume that $\bar{f}(1) = \bar{g}(1)$. Then by Theorem 2.2, $f \simeq g$ rel $\{0,1\}$. So χ is injective. Moreover, let $f(s) = (\cos(2k\pi s), \sin(2k\pi s))$, where $k \in \mathbb{Z}$. Then $\chi([f]) = \frac{1}{2\pi}\bar{f}(1) = k$. So χ is also surjective. It remains to prove that χ is a homomorphism. Given two loops f and g. Recall that

$$f * g(s) = \begin{cases} f(2s) & \text{, if } s \in [0, \frac{1}{2}], \\ g(2s-1) & \text{, if } s \in [\frac{1}{2}, 1]. \end{cases}$$

Obviously, $\bar{f * g}(1) = \bar{f}(1) + \bar{g}(1)$. So $\chi([f] * [g]) = \chi([f * g]) = \frac{1}{2\pi}\bar{f * g}(1) = \frac{1}{2\pi}(\bar{f}(1) + \bar{g}(1)) = \chi([f]) + \chi([g])$. Hence χ is a homomorphism. Therefore, χ is an isomorphism, and we complete this proof. □

3 Proof of Theorem 2.2

Before to prove Theorem 2.2, we give a remark and a lemma.

Remark 3.1. *Let f and g be loops. If $f \simeq g$ rel $\{0,1\}$, then there is a homotopy $F(s,t)$ between f and g relative to $\{0,1\}$ with the properties $F(s,0) = f(s)$ and $F(s,1) = g(s)$ for all $s \in [0,1]$, and $F(0,t) = F(1,t) = (1,0)$ for all $t \in [0,1]$. Let $F_t(s) = F(s,t)$. Fix $s_0 \in [0,1]$. Then $F_t(s_0) : [0,1] \to S^1$ is a map with variable t. Moreover, note that for each $t \in [0,1]$, $\bar{F}_t$ makes sence since F_t is a loop. Therefore,* *for each $t \in [0,1]$, we get a value $\bar{F}_t(s_0) \in \mathbb{R}$ for the fixed $s_0 \in [0,1]$. That is, $\bar{F}_t(s_0) : [0,1] \to \mathbb{R}$ is a function with variable t. In the following, we keep in mind $F_t(s_0)$ and $\bar{F}_t(s_0)$ are both maps with variable t for some fixed s_0.*

Lemma 3.1. *Given two loops f and g with $f \simeq g$ rel $\{0,1\}$. Let F_t be described in Remark 3.1. Then $\bar{F}_t(1) : [0,1] \to \mathbb{R}$ is a constant function.*

Proof. Given $t_0 \in [0,1]$. Since F is continuous, $F_t(s)$ is close to $F_{t_0}(s)$ as (s,t) is close to (s,t_0) for each $s \in [0,1]$. So the cumulate length of F_t from 0 to 1 is close to the cumulate length of F_{t_0} from 0 to 1. That is, $\bar{F}_t(1)$ is close to $\bar{F}_{t_0}(1)$. Hence $\bar{F}_t(1)$ is continuous at t_0. Since t_0 is arbitrary in $[0,1]$, $\bar{F}_t(1)$ is a continuous function. Since F_t is a loop for any $t \in [0,1]$, the range of $\bar{F}_t(1)$ is $\{2n\pi \mid n \in \mathbb{Z}\}$. Since $\bar{F}_t(1)$ is a continuous function and $[0,1]$ is connected, there is $n \in \mathbb{Z}$ such that the range of $\bar{F}_t(1)$ is $\{2n\pi\}$. That is, $\bar{F}_t(1)$ is a constant function. So we complete this proof. □

We are on the position to prove Theorem 2.2.

Proof of Theorem 2.2. Given two loops f and g with $f \simeq g$ rel $\{0,1\}$. Let F_t be described in Remark 3.1. Then by Lemma 3.1, $\bar{F}_t(1) : [0,1] \to \mathbb{R}$ is a constant function. Therefore, $\bar{f}(1) = \bar{F}_0(1) = \bar{F}_1(1) = \bar{g}(1)$.

On the other hand, given two loops f and g with $\bar{f}(1) = \bar{g}(1)$. Define $G(s,t) = (1-t)\bar{f}(s) + t\bar{g}(s)$. Then G is a homotopy between $\bar{f}$ and $\bar{g}$ relative to $\{0,1\}$. Define $F(s,t) = (\cos(G(s,t)), \sin(G(s,t)))$. Then by Remark 2.2, F is obviously a homotopy between f and g relative to $\{0,1\}$. That is, $f \simeq g$ rel $\{0,1\}$.

So we complete this proof. □

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