Mod 2 Homology Of $BSO(3)$

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Abstract

By considering $BO(2) \simeq_s BSO(3) \vee Y_2$ when localized prime 2 in Yan's paper [1], we give the general form of special generators of mod 2 homology of Y_2.

1.**Introduction**

In [1], Yan considered the stable fiberation

$$Y_{2m} \longrightarrow BO(2m) \xrightarrow{Bg_{2m}} BSO(2m+1)$$

and the stable map

$$q_{2m} : BSO(2m+1) \longrightarrow BO(2m),$$

such that

$$(q_{2m})^* \circ (B_{g2m})^* : H^*(BO(2n+1); Z/2) \longrightarrow H^*(BO(2n+1); Z/2)$$

is multiplication by $1 (\bmod 2)$, where $BO(2m)$ is the classifying space of $(2m)$-dimensional real orthogonal group, $BSO(2m+1)$ is the classifying space of $(2m+1)$-dimensional real special orthogonal group.

Given $g_{2m} : O(2m) \longrightarrow SO(2m+1)$ by $g_{2m}(\alpha) = \det(\alpha) \oplus \alpha$, then Y_{2m} be the stable fiber of $Bg_{2m} : BO(2m) \longrightarrow BSO(2m+1)$.

By considering the splitting exact sequence,

$$0 \longrightarrow H_*(Y_{2m}; Z/2) \longrightarrow H_*(BO(2m); Z/2) \xrightarrow{B_{g2m}} H_*(BSO(2m{+}1); Z/2) \longrightarrow 0,$$

we know that $H_*(Y_{2m}; Z/2)$ is the kernel of $(Bg_{2m})_*$.

If we can describe the form of generators of $H_*(Y_{2m}; Z/2)$, in other words, we also can describe the form of generators of $H_*(BSO(2m+1); Z/2)$.

Let f_{2n} be factored as

$$f_{2n} : BO(2n) \stackrel{\triangle}{\to} BO(2n) \times BO(2n) \xrightarrow{\det \times 1} BO(1) \times BO(2n) \stackrel{\oplus}{\to} BO(2n+1),$$

where $\triangle$ denotes the diagonal map, det denotes the map which classifies determinant bundle, and $\oplus$ denotes the Whitney sum map. Since $H_*(BO(2n); Z/2)$ is generated by $b_{m_1}^{r_1} b_{m_2}^{r_2} b_{m_3}^{r_3} \ldots\ldots b_{m_k}^{r_k}$ with at most $2n$ factors, where b_i comes from the usual map $RP^\infty \to BO$. In [2], Yan showed that $(f_{2n})_* : H_*(BO(2n)) \to H_*(BO(2n+1))$ by

$$(f_{2n})_*(b_{m_1} b_{m_2} \cdots b_{m_{2n}}) =$$
$$\sum \frac{(i_1 + i_2 + \cdots + i_{2n})!}{i_1! i_2! \cdots i_{2n!}} b_{i_1+i_2+\cdots+i_{2n}} b_{m_1-i_1} b_{m_2-i_2} \cdots b_{m_{2n}-i_{2n}},$$

where $m_j \geq 0$, and the sum is taken over the sequence $(i_1, i_2, \cdots, i_{2n})$.

We have the following commutative diagram

$$\begin{array}{ccc} BO(2n) & \xrightarrow{Bg_{2n}} & BSO(2n+1) \\ & \searrow^{f_{2n}} & \downarrow h_{2n+1} \\ & & BO(2n+1) \end{array},$$

where h_{2n+1} is the usual 2-folds map.

In view of $(h_{2n+1})_* : H_*(BSO(2n)) \to H_*(BO(2n))$ is injective, we can compute $(Bg_{2m})_*$ by computing $(f_{2n})_*$ and pulling one back of $(h_{2n+1})_*$. That is, $H_*(Y_{2m}; Z/2) = \ker(Bg_{2m})_* = \ker(f_{2n})_*$.

We consider

$$(f_{2n})_*(b_r b_s) = \sum \frac{(i+j)!}{i!j!} b_{i+j} b_{r-i} b_{s-j}.$$